# Angular momentum

# commutation rules

$$\left[J_i, J_j\right] = i\varepsilon_{ijk} J_k \qquad \left[J^2, J_i\right] = 0$$

- This rule is general, and is not restricted to the orbital angular momentum of a single particle.
- $J^2$ and $J_3$ are chosen as the compatible observables to be diagonalized simultaneously.

# eigenkets

- The simultaneous eigenkets are called |jm>, and the eigenvalues are defined by

$$J^2|jm\rangle = j(j+1)|jm\rangle \qquad J_3|jm\rangle = m|jm\rangle$$

why? we will see later

# raising and lowering operators

- Define two non-Hermitian operators

$$J_{\pm} = J_1 \pm iJ_2$$

$$J_{+} = J_{-}^{\dagger}$$

- the commutation rules

$$\left[J_{+}, J_{-}\right] = \left[J_1 + iJ_2, J_1 - iJ_2\right] = -2i\left[J_1, J_2\right] = 2J_3$$

$$\left[J_{+}, J_3\right] = \left[J_1 + iJ_2, J_3\right] = -iJ_2 - J_1 = -J_{+}$$

$$\left[J_{-}, J_3\right] = \left[J_1 - iJ_2, J_3\right] = -iJ_2 + J_1 = J_{-}$$

$$\left[J_{\pm}, J^2\right] = \left[J_1 \pm iJ_2, J^2\right] = 0$$

$$J_+J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J_1^2 + J_2^2 - i[J_1, J_2] = J^2 - J_3^2 + J_3$$

$$J_-J_+ = J_+J_- - [J_+, J_-] = J^2 - J_3^2 - J_3$$

- Because $[J_\pm, J^2] = 0$ $\quad |\alpha_\pm\rangle = J_\pm|jm\rangle$

$$J^2|\alpha_\pm\rangle = J_\pm J^2|jm\rangle = j(j+1)|\alpha_\pm\rangle$$

$$J_3|\alpha_\pm\rangle = \{J_\pm J_3 - [J_\pm, J_3]\}|jm\rangle = \{J_\pm J_3 \pm J_\pm\}|jm\rangle$$
$$= (m \pm 1)J_\pm|jm\rangle = (m \pm 1)|\alpha_\pm\rangle$$

- Thus $J_\pm|jm\rangle$ are eigenkets of $J^2$ with eigenvalue $j(j+1)$ and $J_3$ with eigenvalues $m\pm1$

$$J_\pm|jm\rangle = a_\pm|jm \pm 1\rangle$$

- To determine $a_\pm$

$$\langle jm | J_\mp J_\pm | jm \rangle = |a_\pm|^2 \langle jm \pm 1 | jm \pm 1 \rangle$$
$$= \langle jm | J^2 - J_3^2 \mp J_3 | jm \rangle$$
$$= j(j+1) - m(m \pm 1)$$

- We choose phase convention so that

$$|a_\pm|^2 = j(j+1) - m(m \pm 1)$$

$$a_\pm = \sqrt{j(j+1) - m(m \pm 1)}$$
$$a_+ = \sqrt{(j-m)(j+m+1)}$$
$$a_- = \sqrt{(j+m)(j-m+1)}$$

# the constraint on *m*

$$\langle jm | J^2 | jm \rangle = j(j+1) \geq 0$$

$$\langle jm | J_1^2 + J_2^2 | jm \rangle = j(j+1) - m^2 \geq 0$$

$$m^2 \leq j(j+1)$$

- The right-hand side of this last equation must vanish when m reaches its limiting values

$$0 = \sqrt{j(j+1) - m_>(m_> + 1)} \qquad m_> = j$$

$$0 = \sqrt{j(j+1) - m_<(m_< - 1)} \qquad m_< = -j$$

this is why we use *j*(*j*+1)

# the constraint on $j$

- Assume $m_0$ is an allowed eigenvalue, $k$ some integer, and consider $(J_+)^k|jm_0>$. This is proportional to $|jm_0+k>$, which must for some $k$ satisfy the condition $m_0+k=m_>$.
- By the same argument but using $(J_-)^l|jm_0>$, it must be that $m_0-l=m_<=-m_>$, where $l$ is another integer.
- Hence $k + l = 2m_> = 2j$ must be an integer.

# Spectrum

- The eigenvalues of $J^2$ are $j(j + 1)$, where $j = 0, 1/2, 1, 3/2, ---$
- Given $j$, there are $2j + 1$ eigenstates of $J_3$ with eigenvalues $m = j, j-1, \ldots, -j+1, -j$.

- The whole space can be constructed once one of its members is known

$$|jm\rangle = \left[\frac{(j+m)!}{2j!(j-m)!}\right]^{\frac{1}{2}} (J_-)^{j-m} |jj\rangle$$

$$= \left[\frac{(j-m)!}{2j!(j+m)!}\right]^{\frac{1}{2}} (J_+)^{j+m} |j-j\rangle$$

# int j vs. half-int j

- The state under a rotation about the z-axis

$$D|jm\rangle = e^{-i\theta J_3}|jm\rangle = e^{-im\theta}|jm\rangle$$

- For a complete rotation through θ=2π, this state will not change if *j* is an integer, but will change sign if *j* is half-integer.
- States of integer angular momentum are single-valued under rotation, whereas states of half-integer- angular momentum are double-valued under rotation.

# Introduction of L

- using p-operators

$$L_x = \frac{\hbar}{i}\left( y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$L_y = \frac{\hbar}{i}\left( z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$L_z = \frac{\hbar}{i}\left( x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

- In spherical coordinate

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

$$L_x = i\hbar\left( \sin\varphi\frac{\partial}{\partial \theta} + \cot\theta\cos\varphi\frac{\partial}{\partial \phi} \right)$$

$$L_y = i\hbar\left( -\cos\varphi\frac{\partial}{\partial \theta} + \cot\theta\sin\varphi\frac{\partial}{\partial \phi} \right)$$

# coordinate transformation

- the operators in spherical coordinate
- We express the components of angular momentum to be

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left( x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = x^2 + y^2 + z^2$$

$$\cos\theta = \frac{z}{r}$$

$$\tan\phi = \frac{y}{x}$$

# calculation

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \cos\theta}{\partial x}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial x}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial x} = \frac{xz}{r^3\sin\theta} = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \cos\theta}{\partial y}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial y}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial y} = \frac{yz}{r^3\sin\theta} = \frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial \theta}{\partial z} = \frac{\partial \cos\theta}{\partial z}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = -\frac{1}{r\sin\theta} + \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial z} = -\frac{x^2+y^2}{r^3\sin\theta} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \tan\phi}{\partial x}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\cos^2\phi\frac{y}{x^2} = -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \tan\phi}{\partial y}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \cos^2\phi\frac{1}{x} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial z} = 0$$

# L operators in $\theta$, $\varphi$

$$\frac{i}{\hbar}L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$= y\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) - z\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)$$

$$= y\left(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - z\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \cos^2\phi\frac{1}{x}\frac{\partial}{\partial\phi}\right)$$

$$= -\left(y\frac{\sin\theta}{r} + z\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} - \cos^2\phi\frac{z}{x}\frac{\partial}{\partial\phi}$$

$$= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\frac{i}{\hbar}L_y = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

all these calculations can be done by sympy

$$\begin{aligned}
\frac{i}{\hbar}L_y &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\
&= z\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) - x\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) \\
&= z\left(\frac{x}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \cos^2\phi\frac{y}{x^2}\frac{\partial}{\partial\phi}\right) - x\left(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \left(x\frac{\sin\theta}{r} + z\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \cos^2\phi\frac{yz}{x^2}\frac{\partial}{\partial\phi} \\
&= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}
\end{aligned}$$

# $L^2$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 = \hbar^2 \left[ -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{\hbar^2 r^2}L^2$$

# Schrodinger representation

- the Schrodinger representation of the angular momenta,

$$L_3 \to \frac{1}{i}\frac{\partial}{\partial\phi}$$

$$L_\pm \to \pm e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- The wave function in this expression is called a spherical harmonic:

$$\langle\theta\phi|\,jm\rangle = Y_{lm}(\theta\phi)$$

- the eigenvalue equation for $L_3$ is

$$\left(L_3 - m\right)Y_{lm}\left(\theta\phi\right) = \left(\frac{1}{i}\frac{\partial}{\partial\phi} - m\right)Y_{lm}\left(\theta\phi\right) = 0$$

- The solution is

$$Y_{lm}\left(\theta\phi\right) = e^{im\phi}y_{lm}\left(\theta\right)$$

- The raising and lowering operators give zero when acting on the states with *m* = *l* and *m* = -*l*,

$$L_+Y_{l,l}(\theta\phi) = L_-Y_{l,-l}(\theta\phi) = 0$$

$$\left(\frac{d}{d\theta} - l\cot\theta\right)y_{l,\pm l}(\theta) = 0$$

$$\frac{y'_{l,\pm l}(\theta)}{y_{l,\pm l}(\theta)} = l\frac{\cos\theta}{\sin\theta}$$

- the solution is

$$y_{l,\pm l}(\theta) = c_l \sin^l(\theta)$$

- This result eliminates half-integer values of *l* as eigenvalues of orbital angular momentum.

- All the the spherical harmonics for a given l can be computed from $Y_{ll}$ by repeated use of $L_{\pm}$

$$L_{-}Y_{lm}(\theta\phi) = \sqrt{l(l+1) - m(m+1)}\,Y_{lm-1}(\theta\phi)$$

- The $Y_{lm}$ can be expressed as

$$Y_{lm}(\theta\phi) = \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}}\,(L_{-})^{l-m}\,Y_{ll}(\theta\phi)$$

- $Y_{lm}$ has a function form $Y_{lm} = e^{im\phi} f(\theta)$

$$L_{-} e^{im\phi} f(\theta) = -e^{i(m-1)\phi}\left(\frac{d}{d\theta} + m\cot\theta\right) f(\theta)$$

$$= e^{i(m-1)\phi} \sin^{1-m}\theta \frac{d}{d\cos\theta}\left[f(\theta)\sin^{m}\theta\right]$$

- Once again, we can note $L_{-} e^{im\phi} f(\theta) = e^{i\lambda\phi} g(\theta)$
- The repetition leads to

$$\left(L_{-}\right)^{k} e^{im\phi} f(\theta) = e^{i(m-k)\phi} \sin^{k-m}\theta \frac{d^{k}}{d^{k}\cos\theta}\left[f(\theta)\sin^{m}\theta\right]$$

- The normalization of the wavefunction

$$y_{l,\pm l}(\theta) = c_l \sin^l(\theta)$$

$$1 = \langle ll|ll\rangle = |c_l|^2 y_{l,\pm l}(\theta) = |c_l|^2 \int d\phi \int d\cos\theta \sin^{2l}\theta$$

$$= |c_l|^2 4\pi \frac{\left(2^l l!\right)^2}{(2l+1)!}$$

- with the traditional phase convention

$$c_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}$$

$$Y_{lm} = \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}} \left(L_{-}\right)^{l-m} Y_{ll}$$

$$= (-1)^{l} \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}} c_{l} e^{im\phi} \sin^{-m}\theta \frac{d^{l-m}}{d^{l-m}\cos\theta} \left(\sin^{2l}\theta\right)$$

$$= \frac{(-1)^{l}}{2^{l} l!} \sqrt{\frac{(l+m)!}{(l-m)!}} \sqrt{\frac{2l+1}{4\pi}} e^{im\phi} \sin^{-m}\theta \frac{d^{l-m}}{d^{l-m}\cos\theta} \left(\sin^{2l}\theta\right)$$

- when $m$ = 0 it reduce to the Legendre polynomial

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

- For $m$ > 0, the spherical harmonics are proportional to associated Legendre function

$$Y_{lm} = (-1)^m \sqrt{\frac{(l+m)!}{(l-m)!} \frac{2l+1}{4\pi}} e^{im\phi} P_l^m(\cos\theta)$$

- For negative $m$

$$Y_{lm}^* = (-1)^m Y_{l,-m}$$

- The spherical harmonics form a complete, single-valued, and orthonormal set on the unit sphere.

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta\phi)Y^{*}_{lm}(\theta'\phi') = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta}$$

- addition theorem

$$P_l(n_1 \cdot n_2) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta_1\phi_1)Y^{*}_{lm}(\theta_2\phi_2)$$

- for $l$ = 1

$$Y_{1,\pm 1}(\theta\phi) = \sqrt{\frac{3}{8\pi}} e^{\pm\phi} \sin\theta$$

$$Y_{10}(\theta\phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\left(Y_{11}, Y_{10}, Y_{1,-1}\right) = \sqrt{\frac{3}{4\pi}} \frac{1}{r} \left( \frac{-x_1 - ix_2}{\sqrt{2}}, x_3, \frac{x_1 - ix_2}{\sqrt{2}} \right)$$

$$\begin{pmatrix} Y_{11} \\ Y_{10} \\ Y_{1,-1} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \left( \sqrt{\frac{3}{4\pi}} \frac{1}{r} \right)$$

$$= W^{\dagger} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} N$$

$$W = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

- let R be the matrix that produces the rotated coordinates x when acting on the column vector formed by x'

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} Y_{11}(\mathbf{n}') \\ Y_{10}(\mathbf{n}') \\ Y_{1,-1}(\mathbf{n}') \end{pmatrix} = W^{\dagger} R W \begin{pmatrix} Y_{11}(\mathbf{n}) \\ Y_{10}(\mathbf{n}) \\ Y_{1,-1}(\mathbf{n}) \end{pmatrix} \qquad \mathbf{n} \equiv (\theta\phi)$$

# Spin 1/2

- When j=1/2 it is convenient to replace the 2 x 2 angular momentum matrices by the Pauli matrices

$$J = \frac{1}{2}\sigma$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The raising and lowering operators

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- The Pauli matrices and the unit matrix form a complete set of 2x 2 matrices.

- An arbitrary 2 x 2 matrix M can be written as

$$M = m_0 1 + \sum_i m_i \sigma_i = \begin{pmatrix} m_0 + m_3 & m_1 + i m_2 \\ m_1 - i m_2 & m_0 - m_3 \end{pmatrix}$$

- The commutation relations

$$\left[\sigma_a,\sigma_b\right]=2i\varepsilon_{abc}\sigma_c$$

$$\left\{\sigma_a,\sigma_b\right\}=2\delta_{ab}$$

$$\sigma_a\sigma_b=\delta_{ab}+i\varepsilon_{abc}\sigma_c$$

- quadratic identity

$$\left(\sigma_i A_i\right)\left(\sigma_j B_j\right)=A_i B_i+i\varepsilon_{ijk}\sigma_k A_i B_j$$

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\sigma\cdot(\mathbf{A}\times\mathbf{B})$$

- unitary rotation operator

$$D^{\frac{1}{2}}(\theta) = e^{-i\theta n\cdot J} = e^{-\frac{1}{2}i\theta n\cdot\sigma}$$

$$D^{\dagger}(\delta\theta)\sigma D(\delta\theta) = \left(1 + \frac{1}{2}i\delta\omega\cdot\sigma\right)\sigma\left(1 - \frac{1}{2}i\delta\omega\cdot\sigma\right)$$

$$D^{\dagger}(\delta\theta)\sigma_i D(\delta\theta) = \sigma_i + \frac{1}{2}i\delta\omega_j\left(\sigma_j\sigma_i - \sigma_i\sigma_j\right)$$

$$= \sigma_i + \varepsilon_{ijk}\delta\omega_j\sigma_k$$

$$D^{\dagger}(\delta\theta)\sigma D(\delta\theta) = \sigma + \delta\omega\times\sigma$$

- σ transforms like a vector under rotations generated by the angular momentum operator J= σ/2

- If the components of A commute among themselves

$$(\sigma \cdot A)^2 = A^2$$

- for a finite rotation $D(R) = e^{-i\Theta}$

$$\Theta = \frac{1}{2}\theta \mathbf{n} \cdot \sigma \qquad \Theta^{2n} = \left(\frac{1}{2}\theta\right)^{2n} (\mathbf{n} \cdot \sigma)^{2n} = \left(\frac{1}{2}\theta\right)^{2n}$$

$$\Theta^{2n+1} = \left(\frac{1}{2}\theta\right)^{2n+1} \mathbf{n} \cdot \sigma = \Theta \left(\frac{1}{2}\theta\right)^{2n}$$

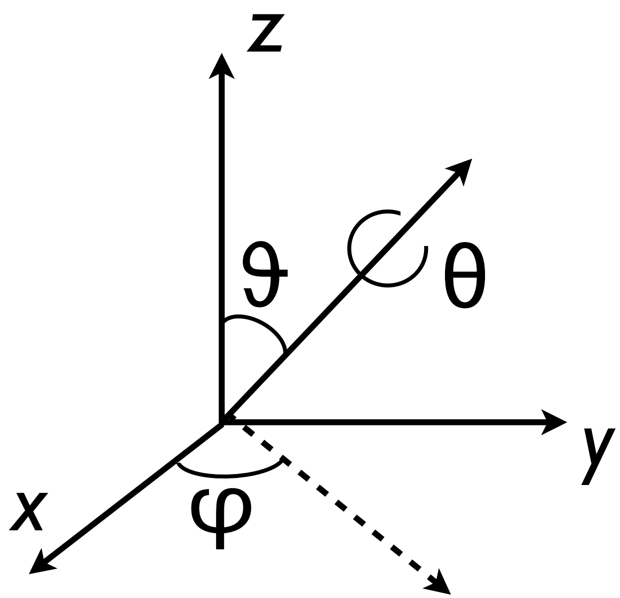
$$e^{-i\Theta} = \sum_n \frac{1}{n!}(-i\Theta)^n = \sum_{n=\text{even}} \frac{1}{n!}\left(\frac{i\theta}{2}\right)^n - i\Theta \sum_{n=\text{odd}} \frac{1}{n!}\left(\frac{i\theta}{2}\right)^{n-1}$$

$$= \cos\frac{\theta}{2} - i\Theta \sin\frac{\theta}{2}$$

- This rotation matrix is double-valued

- Caley-Klein parameters

$$\mathbf{n} = \left(\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta\right)$$




$$D(R) = \cos\frac{\theta}{2}1 - i\sin\frac{\theta}{2}\sum_i n_i\sigma_i$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\cos\vartheta & -i\sin\frac{\theta}{2}\sin\vartheta e^{i\phi} \\ i\sin\frac{\theta}{2}\sin\vartheta e^{i\phi} & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\vartheta \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

- The most general state in the Hilbert space is described by a density matrix which must also be expressible in the form

$$\rho = \frac{1}{2}\left(1 + P\cdot\sigma\right) \qquad \rho^2 = \frac{1}{4}\left(1 + P\cdot\sigma\right)^2 = \frac{1}{4}\left(1 + 2P\cdot\sigma + P^2\right)$$

$$\mathrm{Tr}\rho = 1 \qquad \mathrm{Tr}\rho^2 = \frac{1}{2}\left(1 + P^2\right)$$

- P is called the polarization. |P|<=1 with IPI = 1 being the case of a pure state.

$$\langle\sigma\rangle = \mathrm{Tr}\left(\rho\sigma\right) = P$$

# Addition of angular momenta

- addition of two angular momenta $J_1$ and $J_2$ that belong to distinct, independent degrees of freedom whose commutators vanish
- The state space in question, S'Jjd2' is spanned by products of eigenstates of the separate angular momenta,

$$\left| j_1 m_1 \right\rangle \otimes \left| j_2 m_2 \right\rangle = \left| j_1 j_2 m_1 m_2 \right\rangle$$

- the space dimension

$$d = (2j_1 + 1)(2j_2 + 1)$$

- The total angular momentum operator is $J = J_1 + J_2$
- The objective is to construct simultaneous eigenstates $|jm>$ of $J$ and $J_z$.
- Scalars are invariant to rotations so they are commute to J and $J_z$ . So possible choices are $J_1^2, J_2^2$ and

$$J_1 \cdot J_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$$

- One may chose $J^2, J_z, J_1^2, J_2^2$ as the commuting operators

- Q1 : given $j_1$ and $j_2$, to find the spectrum of permitted values of j
- Ans: j satisfies a triangular inequality reminiscent of ordinary vector addition,

$$j_1 + j_2 \geq j \geq \left| j_1 - j_2 \right|$$

- The allowed values of j differ by integers.
- For a given j there is only one sequence of (2j + 1) states, with m = j,j-1,..., -j, for each allowed value of j.

# Adding Spins 1/2

- The space the addition of two spin is four-dimensional

$$S = \frac{1}{2}(\sigma_1 + \sigma_2)$$

- The allowed values of S are (i) an S = 1 triplet with $M_s$ = 1,0, -1; and (ii) an S = 0 singlet.

- The states with maximum |Ms|=1, are $|+\rangle_1|+\rangle_2$ and $|-\rangle_1|-\rangle_2$
- Using the lowering operator

$$S_{\pm}\left|1M_S\right\rangle = \sqrt{2 - M_S\left(M_S \pm 1\right)}\left|1M_S \pm 1\right\rangle$$

- The final result for the triplet

$$\left|11\right\rangle = \left|+\right\rangle_1\left|+\right\rangle_2$$

$$\left|10\right\rangle = S_{-}\left|11\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+\right\rangle_1\left|-\right\rangle_2 + \left|-\right\rangle_1\left|+\right\rangle_2\right)$$

$$\left|1-1\right\rangle = \left|-\right\rangle_1\left|-\right\rangle_2$$

- The singlet

$$\left|00\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2\right)$$

- Under interchange of the two individual spin projections, the S=1 spin triplet is symmetric, while the S=0 singlet is antisymmetric.

$$S^2 = \frac{1}{2}\left(3+\sigma_1\cdot\sigma_2\right) = S\left(S+1\right) = \begin{cases} 2 & \text{triplet} \\ 0 & \text{singlet} \end{cases}$$

- The projection operators onto the subspaces of definite S are

$$P_s = 1-\frac{S^2}{2} = \frac{1}{4}\left(1-\sigma_1\cdot\sigma_2\right)$$

$$P_t = \frac{S^2}{2} = \frac{1}{4}\left(3+\sigma_1\cdot\sigma_2\right)$$

- operator interchanges the two spins. $P_{12}$

$$P_{12} = -P_s + P_t = \frac{1}{2}\left(1 + \sigma_1 \cdot \sigma_2\right)$$

# Irreducibility

- The $(2j_1 + 1)(2j_2 + 1)$ dimensional space spanned by the product states $|j_1m_1>|j_2m_2>$ can be decomposed into rotationally invariant spaces

$$(2j_1+1)\otimes(2j_2+1)=(2j_1+2j_2+1)\oplus(2j_1+2j_2-1)\cdots\oplus(2|j_1-j_2|+1)$$

- The right side is called the irreducible space. Irreducibility refers to that within each of the latter any ket will, under arbitrary rotations, become a linear combination of all the kets in that same space.

# Clebsch-Gordan Coefficients

$$\left| j_1 j_2 jm \right\rangle = \sum_{m_1 m_2} \left| j_1 m_1 j_2 m_2 \right\rangle \left\langle j_1 m_1 j_2 m_2 \middle| j_1 j_2 jm \right\rangle$$

- the transformation between bases is unitary,

$$\sum_{jm} \left\langle j_1 m_1 j_2 m_2 \middle| jm \right\rangle \left\langle jm \middle| j_1 m_1' j_2 m_2' \right\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\sum_{m_1 m_2} \left\langle jm \middle| j_1 m_1 j_2 m_2 \right\rangle \left\langle j_1 m_1 j_2 m_2 \middle| j' m' \right\rangle = \delta_{jj'} \delta_{mm'}$$

- All CG coefficients can be computed from recursion relations

$$
\begin{aligned}
J_{\mp}\left| jm \right\rangle &= a_{\mp}(jm)\left| jm \mp 1 \right\rangle \\
&= J_{\mp} \sum_{m_1 m_2} \left| m_1 m_2 \right\rangle \left\langle m_1 m_2 \middle| jm \right\rangle \\
&= \sum_{m_1 m_2} a_{\mp}(j_1 m_1) \left| m_1 \mp 1, m_2 \right\rangle \left\langle m_1 m_2 \middle| jm \right\rangle \\
&\quad + \sum_{m_1 m_2} a_{\mp}(j_2 m_2) \left| m_1, m_2 \mp 1 \right\rangle \left\langle m_1 m_2 \middle| jm \right\rangle
\end{aligned}
$$

$$
a_{\pm}(jm) = \sqrt{j(j+1) - m(m \pm 1)}
$$

$$
\begin{aligned}
a_{\mp}(jm)\left\langle m_1 m_2 \middle| jm \mp 1 \right\rangle &= a_{\mp}(j_1 m_1 \pm 1)\left\langle m_1 \pm 1, m_2 \middle| jm \right\rangle \\
&\quad + a_{\mp}(j_2 m_2 \pm 1)\left\langle m_1, m_2 \pm 1 \middle| jm \right\rangle \\
&= a_{\pm}(j_1 m_1)\left\langle m_1 \pm 1, m_2 \middle| jm \right\rangle \\
&\quad + a_{\pm}(j_2 m_2)\left\langle m_1, m_2 \pm 1 \middle| jm \right\rangle
\end{aligned}
$$

- when m=j

$$0 = a_-(j_1 m_1)\langle m_1 - 1, m_2 | jj\rangle + a_-(j_2 j - m_1)\langle m_1, m_2 - 1 | jj\rangle$$

- $m_2$=j-$m_1$

$$a_-(j_1 m_1)\langle m_1 - 1, j - m_1 + 1 | jj\rangle = -a_-(j_2 j - m_1)\langle m_1, j - m_1 | jj\rangle$$

- When $m_1$=$j_1$

$$a_-(j_1 j_1)\langle j_1 - 1, j - j_1 + 1 | jj\rangle = -a_-(j_2 j - j_1)\langle j_1, j - j_1 | jj\rangle$$

$$\langle j_1 - 1, j - j_1 + 1 | jj\rangle = \frac{-a_-(j_2 j - j_1)}{a_-(j_1 j_1)}\langle j_1, j - j_1 | jj\rangle$$