Transmission lines

Distributed parameter model

Harmonic waves on TL

Terminated TL

The Smith chart

Time domain analysis

S parameters

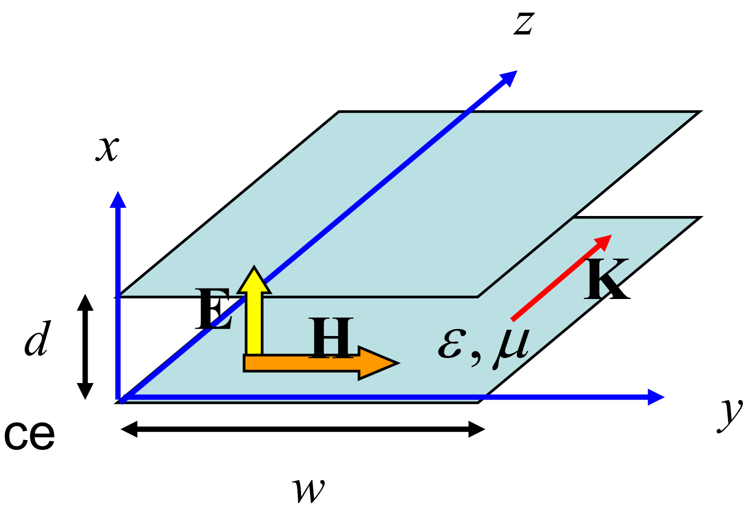
Transmission line equations

The EM field inside the parallel-plate waveguide can be

$$E_x(z,t) = E_0 \cos(kz - \omega t)$$

$$H_y(z,t) = H_0 \cos(kz - \omega t)$$

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}}$$ called the intrinsic impedance



The surface charge density on the plate obeys

$$\sigma = \varepsilon E_\perp = \varepsilon E_x(z,t) = \varepsilon E_0 \cos(kz - \omega t)$$

$$E_\perp = \frac{\sigma}{\varepsilon}$$

The surface current density on the plate obeys

$$\mathbf{K} = H_y(z,t)\hat{\mathbf{z}} = H_0 \cos(kz - \omega t)\hat{\mathbf{z}}$$

$$\mathbf{H}_\parallel = \mathbf{n} \times \mathbf{K}$$

Clearly, the EM fields satisfies the charge conservation on the plate, namely,

$$\nabla \cdot \mathbf{K} + \frac{\partial \sigma}{\partial t} = 0$$

Transmission line equations

One may write down the differential equations for the EM fields

$$\frac{\partial}{\partial z} E_x(z,t) = -\mu \frac{\partial}{\partial t} H_y(z,t)$$

$$\frac{\partial}{\partial z} H_y(z,t) = -\varepsilon \frac{\partial}{\partial t} E_x(z,t)$$

$$V(z,t) = E_x(z,t)\, d$$

$$I(z,t) = K_z(z,t)\, w = H_y(z,t)\, w$$

$$\frac{\partial}{\partial z} V(z,t) = -\mu \frac{d}{w} \frac{\partial}{\partial t} I(z,t) = -L \frac{\partial}{\partial t} I(z,t)$$

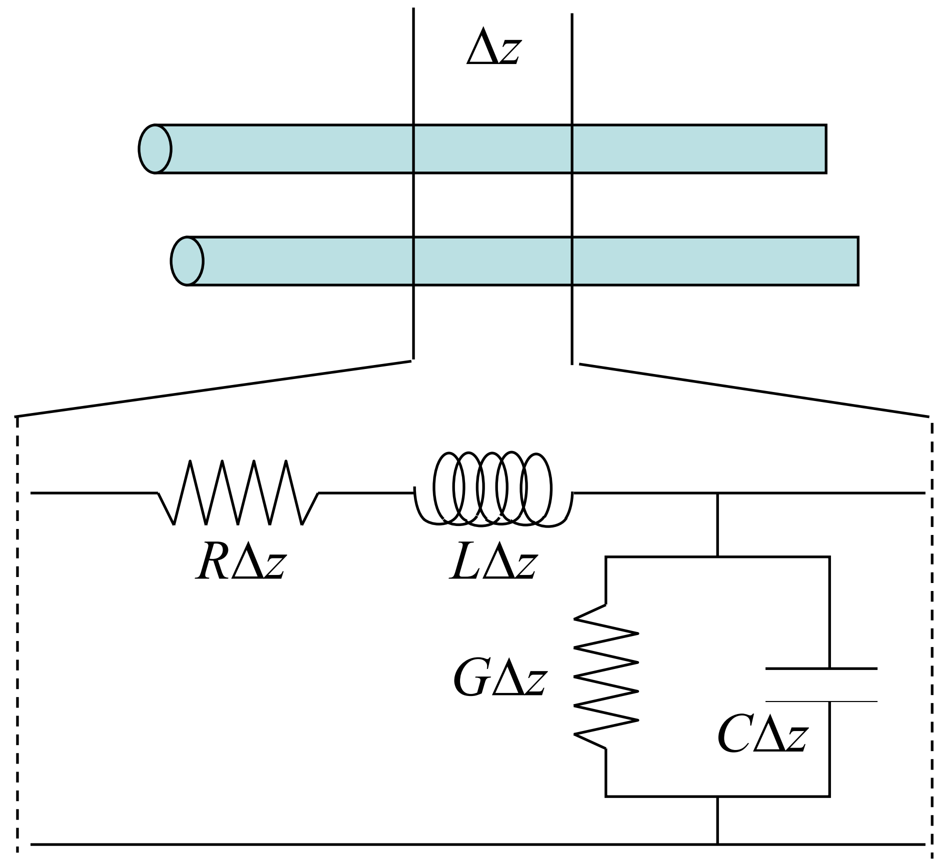
$$\frac{\partial}{\partial z} I(z,t) = -\varepsilon \frac{w}{d} \frac{\partial}{\partial t} V(z,t) = -C \frac{\partial}{\partial t} V(z,t)$$

One may define the capacitance(per unit length) and inductance (per unit length) by

$$L = \mu \frac{d}{w}$$

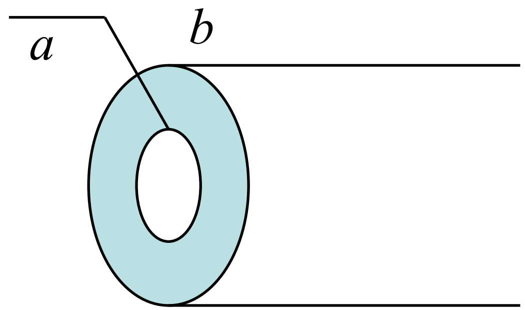
$$C = \varepsilon \frac{w}{d}$$

Distributed-parameter model



Coaxial cables

For coaxial cables, the parameters are



Shunt conductance
$$G = \frac{2\pi\sigma_d}{\ln(b/a)}$$

Shunt capacitance
$$C = \frac{2\pi\varepsilon}{\ln(b/a)}$$

series inductance
$$L = \frac{\mu \ln(b/a)}{2\pi}$$

series resistance
$$R = \frac{1}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)\sqrt{\frac{\mu\omega}{2\sigma_c}}$$

σ_c: conductivity of conductors
σ_d: conductivity of dielectrics

Telegraphist's equations

Applying Kirchhoff's voltage law, we have

$$v(z,t) - v(z+\Delta z,t) = i(z,t)R\Delta z + L\Delta z \frac{\partial i(z,t)}{\partial t}$$

$$\Rightarrow \quad -\frac{\partial v(z,t)}{\partial z} = i(z,t)R + L\frac{\partial i(z,t)}{\partial t}$$

Applying Kirchhoff's current law, we have

$$i(z,t) - i(z+\Delta z,t) = v(z+\Delta z,t)G\Delta z + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

$$\Rightarrow \quad -\frac{\partial i(z,t)}{\partial z} = v(z,t)G + C\frac{\partial v(z,t)}{\partial t}$$

Distributed parameter model

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Harmonic waves on transmission lines

The telegraphist' s equations are coupled space-time dependent differential equations

Consider sinusoidal time-dependent solutions:

$$v(z,t) = V(z)\cos(\omega t + \phi)$$

In which V(z) is a time-independent function

$$v(z,t) = \mathrm{Re}\left[V(z)e^{i\omega t}\right]$$

The phase factor ϕ can be absorbed in V(z)

$$i(z,t) = \mathrm{Re}\left[I(z)e^{i\omega t}\right]$$

Time-independent telegraphist’ s equations

$$-\frac{\partial v(z,t)}{\partial z} = i(z,t)R + L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial V(z)}{\partial z} = -(R + i\omega L)I(z)$$

$$-\frac{\partial i(z,t)}{\partial z} = v(z,t)G + C\frac{\partial v(z,t)}{\partial t}$$

$$\frac{\partial I(z)}{\partial z} = -(G + i\omega C)V(z)$$

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + i\omega L)(G + i\omega C)V(z)$$

$$= -\gamma^2 V(z)$$

The solutions are $V(z) = V_1 e^{-\gamma z} + V_2 e^{\gamma z}$

Traveling wave equations

In general, γ is a complex and can be denoted by $\gamma = \alpha + \beta i$

The parameter α describes the signal loss, while the parameter β describes
The signal propagation along the transmission line

$$v(z,t) = \mathrm{Re}\left[V(z)e^{i\omega t}\right] = V_1 e^{-\alpha z}\cos(\omega t - \beta z) + V_2 e^{\alpha z}\cos(\omega t + \beta z)$$

Traveling wave to +z   Traveling wave to -z

The current satisfies the similar differential equation, so the solutions are

$$i(z,t) = I_1 e^{-\alpha z}\cos(\omega t - \beta z) + I_2 e^{\alpha z}\cos(\omega t + \beta z)$$

Characteristic impedance

The characteristic impedance of a transmission line is defined by

$$Z = \frac{V_1}{I_1}$$

From $$\frac{\partial V(z)}{\partial z} = -(R + i\omega L) I(z)$$

$$\Rightarrow \quad -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z} = -(R + i\omega L)\left(I_1 e^{-\gamma z} + I_2 e^{\gamma z}\right)$$

For all *z* $$\gamma V_1 e^{-\gamma z} = (R + i\omega L) I_1 e^{-\gamma z}$$

$$Z = \frac{V_1}{I_1} = \frac{(R + i\omega L)}{\gamma} = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

For wave traveling to the –z direction $Z = \frac{-V_2}{I_2}$

Lossless line

If α is zero, the wave travels losslessly.

the lossless conditions can be described by

$$R \ll \omega L$$

$$G \ll \omega C$$

In this case, the propagation parameters of the wave becomes

$$\alpha = 0$$

$$\beta = \omega\sqrt{LC}$$

The propagation speed of the wave is $v = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$

and the characteristic impedance is $Z = \sqrt{\dfrac{L}{C}}$

Characteristic impedance of coaxial cables

$$C = \frac{2\pi\varepsilon}{\ln(b/a)} \qquad L = \frac{\mu \ln(b/a)}{2\pi}$$

$$Z = \sqrt{\frac{L}{C}} = \left(\frac{1}{2\pi}\right)^2 \frac{\mu}{\varepsilon} \ln\left(\frac{2b}{a}\right)$$

The wave speed is $$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

We note that the wave speed satisfies this equation for all kinds of transmission lines. One can prove it by using relativity

Power transmission (lossless)

The average power at any point z along the line:

$$P_{ave}(z) = \frac{1}{2}\mathrm{Re}\left[V_s I_s^*\right]$$

On a lossless line, $V_s = V_1 e^{-i\beta z} = |V_1| e^{i\phi} e^{-i\beta z}$

The current is in-phase with V $\quad I_s = |I_1| e^{i\phi} e^{-i\beta z} = \frac{|V_1|}{Z_0} e^{i\phi} e^{-i\beta z}$

$$P_{ave}(z) = \frac{1}{2}\mathrm{Re}\left[|V_1| e^{i\phi} e^{-i\beta z} \frac{|V_1|}{Z_0} e^{-i\phi} e^{+i\beta z}\right] = \frac{1}{2}\frac{|V_1|^2}{Z_0}$$

Power transmission (lossy)

On a lossy line, the current is no longer in-phase with V.
In general, the impedance is complex

$$Z_0 = \left|Z_0\right| e^{i\theta}$$

$$V_s = \left|V_1\right| e^{i\phi} e^{-\alpha z} e^{-i\beta z}$$

$$I_s = \frac{\left|V_1\right|}{\left|Z_0\right|} e^{-i\theta} e^{i\phi} e^{-\alpha z} e^{-i\beta z}$$

$$\Rightarrow \quad P_{ave}(z) = \frac{1}{2} \frac{\left|V_1\right|^2}{Z_0} e^{-2\alpha z} \cos\theta$$

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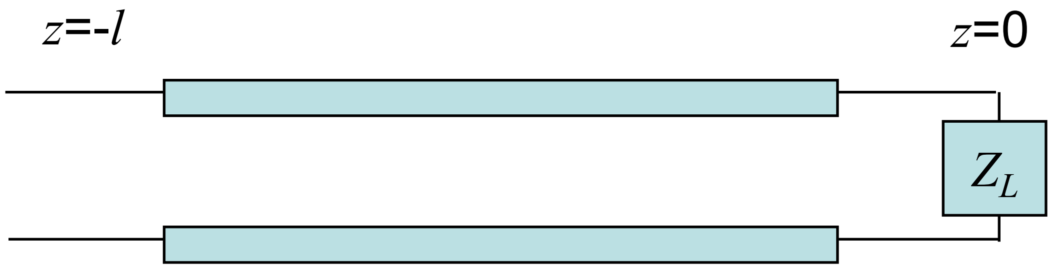
Reflection at a load



$$V(z,t) = \mathrm{Re}\left[V_0\left(e^{i\omega t-\gamma z} + \Gamma_L e^{i\omega t+\gamma z}\right)\right]$$

$$I(z,t) = \mathrm{Re}\left[\frac{V_0}{Z_0}\left(e^{i\omega t-\gamma z} - \Gamma_L e^{i\omega t+\gamma z}\right)\right]$$

Γ: reflection coefficient

Z_0 : the impedance of the transmission line

At z=0, V and I satisfy

$$Z_L = \frac{V(0,t)}{I(0,t)} = Z_0\frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\Rightarrow \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad |\Gamma_L| \le 1$$

For an open circuit, $Z_L \to \infty$ $\Gamma_L = 1$

For a short circuit, $Z_L = 0$ $\Gamma_L = -1$

For a matched load $Z_L = Z_0$ $\Gamma_L = 0$

generalized reflection coefficient

Consider the voltage on any point at z

$$V(z,t) = \mathrm{Re}\left[V_0\left(e^{i\omega t-\gamma z} + \Gamma_L e^{i\omega t+\gamma z}\right)\right] = \mathrm{Re}\left[V_0 e^{i\omega t-\gamma z}\left(1+\Gamma_L e^{2\gamma z}\right)\right]$$

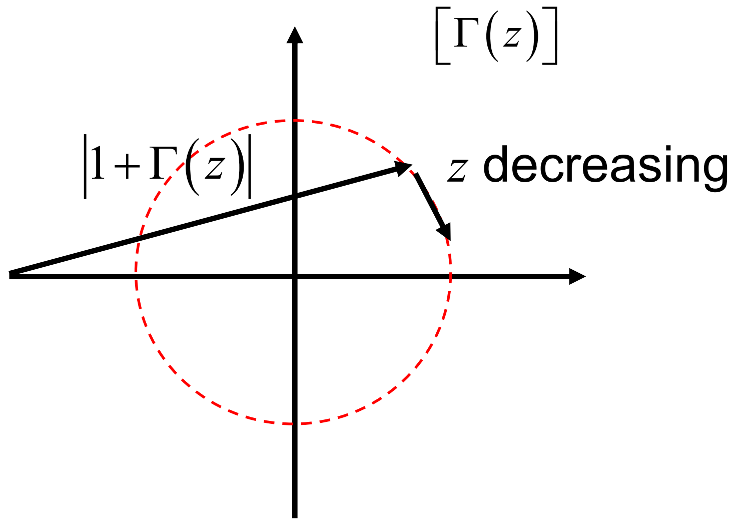
$$\Gamma(z) = \Gamma_L e^{2\gamma z}$$

Called the generalized reflection coefficient

$$\left|\Gamma(z)\right| = \left|\Gamma_L\right| \le 1$$

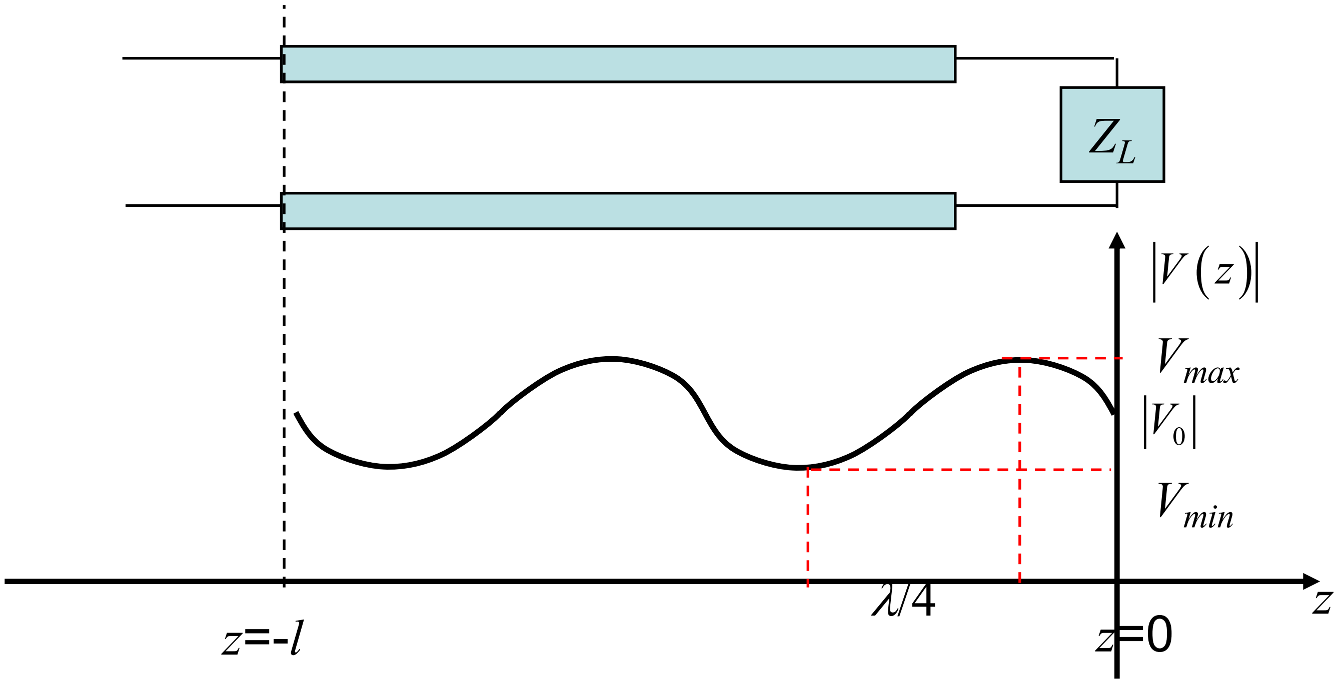
The magnitude of the voltage is a function of z

$$\left|V(z)\right| = \left|V_0\right|\left|1+\Gamma(z)\right|$$



voltage standing wave ratio(VSWR)

Plot the voltage on the transmission line, one get a voltage standing wave pattern



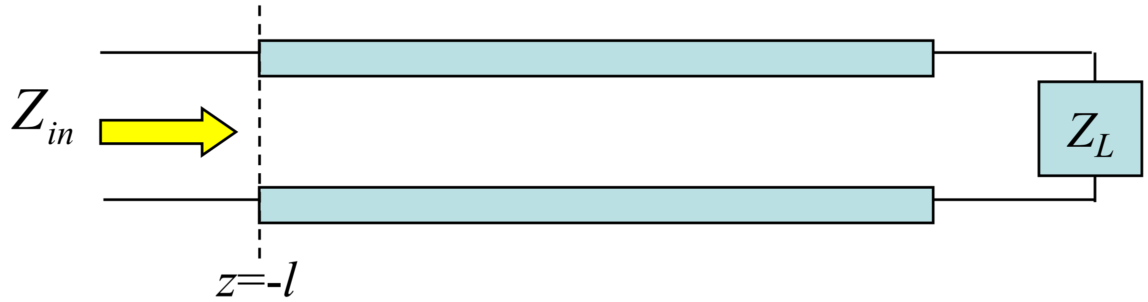
the maximum voltage $V_{\max} = |V_0|(1+|\Gamma_L|)$

the minimum voltage $V_{\min} = |V_0|(1-|\Gamma_L|)$

The voltage standing wave ratio(VSWR) is defined as $VSWR = \dfrac{V_{\max}}{V_{\min}} = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|}$

Input impedance

At z=-l, the ratio of V to I is known as the input impedance



$$Z_{in} = \frac{V_s(-l)}{I_s(-l)} = Z_0 \frac{e^{\gamma l} + \Gamma_L e^{-\gamma l}}{e^{i\gamma l} - \Gamma_L e^{-\gamma l}}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= Z_0 \frac{(Z_L + Z_0)e^{\gamma l} + (Z_L - Z_0)e^{-\gamma l}}{(Z_L + Z_0)e^{\gamma l} - (Z_L - Z_0)e^{-\gamma l}}$$

$$= Z_0 \frac{Z_L \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_0 \cosh(\gamma l) + Z_L \sinh(\gamma l)}$$

$$= Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)}$$

Input impedance for lossless TL

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh\left(\gamma l\right)}{Z_0 + Z_L \tanh\left(\gamma l\right)}$$

$$= Z_0 \frac{Z_L + iZ_0 \tan\left(\beta l\right)}{Z_0 + iZ_L \tan\left(\beta l\right)}$$

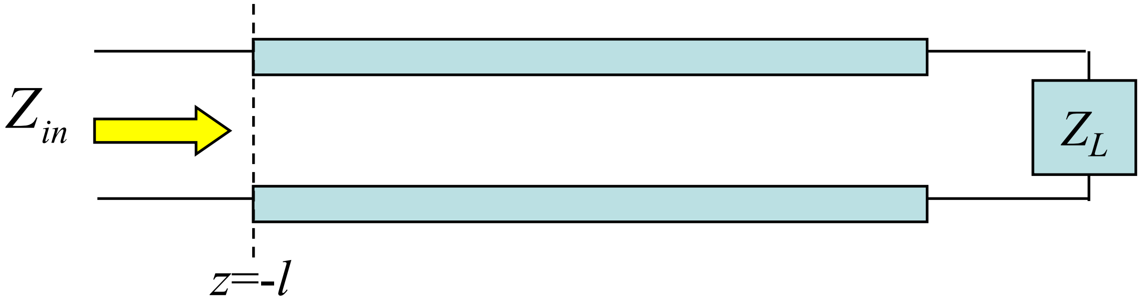
$$\beta l = 0$$

$$Z_{in} = Z_L$$

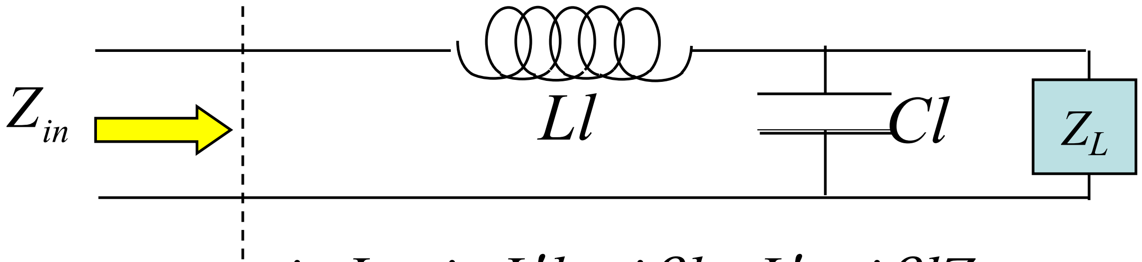
$$\beta l = \frac{\pi}{2}$$

$$Z_{in} = \frac{Z_0^2}{Z_L}$$

Lumped element model



$$Z_{in} = Z_0 \frac{Z_L + iZ_0 \tan(\beta l)}{Z_0 + iZ_L \tan(\beta l)}$$



Lumped element model

$$i\omega L = i\omega L' l = i\beta l v_p L' = i\beta l Z_0$$

$$\frac{1}{i\omega C} = \frac{1}{i\omega C' l} = \frac{1}{i\beta l v_p C'} = \frac{Z_0}{i\beta l}$$

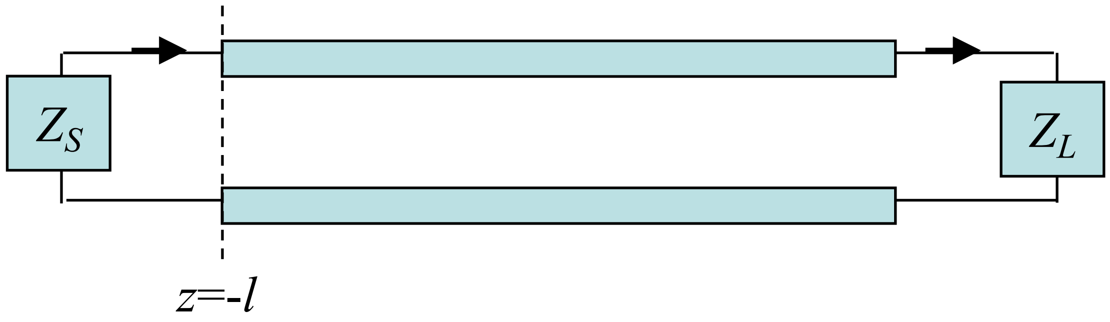
$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$v_p = \sqrt{\frac{1}{L'C'}}$$

$$Z_{in} = i\beta l Z_0 + \frac{1}{\frac{i\beta l}{Z_0} + \frac{1}{Z_L}} = Z_0 \left[i\beta l + \frac{Z_L}{Z_0 + iZ_L \beta l} \right]$$

$$= Z_0 \left[\frac{Z_L + iZ_0 \beta l - Z_L \beta^2 l^2}{Z_0 + iZ_L \beta l} \right]$$

Transmission line resonators



$$V(z,t) = \mathrm{Re}\left[V_0\left(e^{i\omega t-\gamma z} + \Gamma_L e^{i\omega t+\gamma z}\right)\right]$$

$$I(z,t) = \mathrm{Re}\left[\frac{V_0}{Z_0}\left(e^{i\omega t-\gamma z} - \Gamma_L e^{i\omega t+\gamma z}\right)\right]$$

At z=0, V and I satisfy

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

At z=-l, V and I satisfy $\left(e^{i\beta l} + \Gamma_L e^{-i\beta l}\right) = -\frac{Z_S}{Z_0}\left(e^{i\beta l} - \Gamma_L e^{-i\beta l}\right)$

$$\Gamma_L e^{-2i\beta l} = -\frac{Z_S + Z_0}{Z_S - Z_0} = -\frac{1}{\Gamma_S} \Rightarrow e^{2i\beta l} = \Gamma_L \Gamma_S$$

Resonate modes

For short-circuited lines $\Gamma_L = \Gamma_S = -1$

$\Rightarrow$ $e^{2i\beta l} = 1$ $\beta = \frac{n\pi}{l}$ $\omega = \frac{n\pi}{l\sqrt{LC}}$

$\Rightarrow$

$$V(z) = V_0 \sin\frac{n\pi z}{l}$$

$$I(z) = i\frac{V_0}{Z_0}\cos\frac{n\pi z}{l}$$

A small load will result in energy loss : $Z_L = R_L$

$\gamma = \alpha + i\beta$ $e^{2i\beta l}e^{2\alpha l} = \Gamma_L\Gamma_S$

At resonate frequencies $\Gamma_L\Gamma_S = e^{2\alpha l} \simeq 1 + 2\alpha l$

$$\Gamma_S\Gamma_L = -\frac{R_L - Z_0}{R_L + Z_0} \simeq 1 - 2\frac{R_L}{Z_0} \Rightarrow \alpha = -\frac{R_L}{lZ_0}$$

Decay rate $\alpha v = -\frac{R_L}{Z_0}\frac{1}{\sqrt{LC}} = -\frac{R_L}{L}$

Distributed parameter model

Harmonic waves on TL

Terminated TL

The Smith chart

Microwave engineering

Smith chart

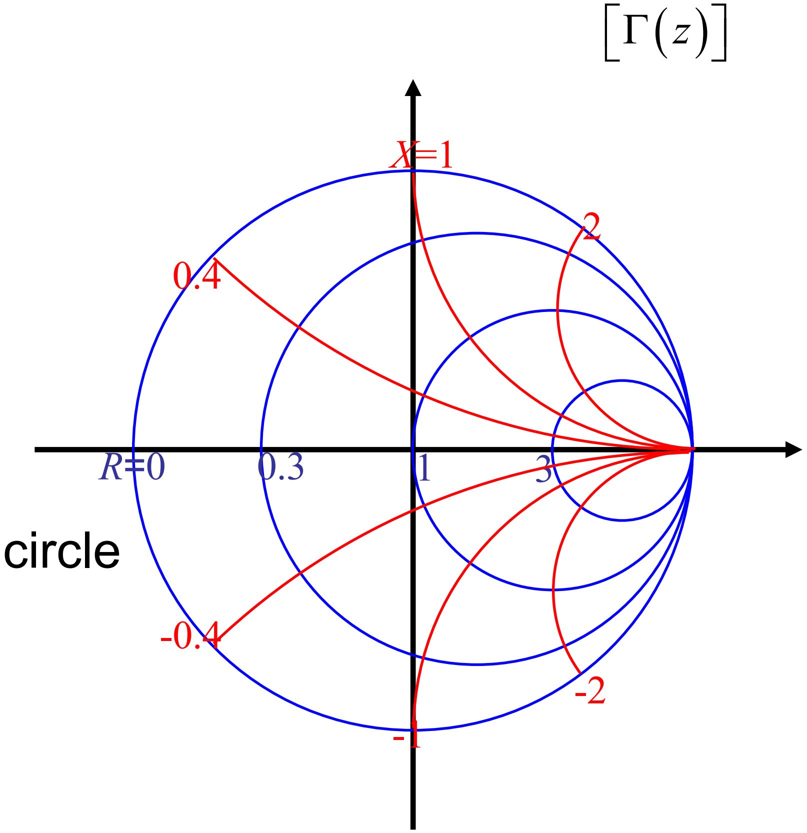
Consider the complex impedance on any point at z

$$Z(z)=\frac{V(z,t)}{I(z,t)}=\frac{V_0\left(e^{i(\omega t-\gamma z)}+\Gamma_L e^{i(\omega t+\gamma z)}\right)}{\frac{V_0}{Z_0}\left(e^{i(\omega t-\gamma z)}-\Gamma_L e^{i(\omega t+\gamma z)}\right)}=Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

We are able to determine the impedance from the complex Γ-plane.

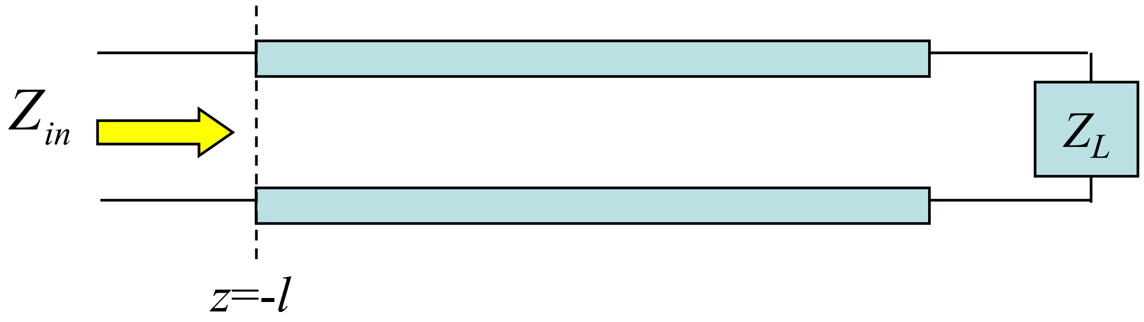
Note that $|\Gamma(z)|=|\Gamma_L|\leq 1$, and one only has to consider the region within the unit circle

$$Z(z)=R+iX$$



unit circle

The use of Smith chart



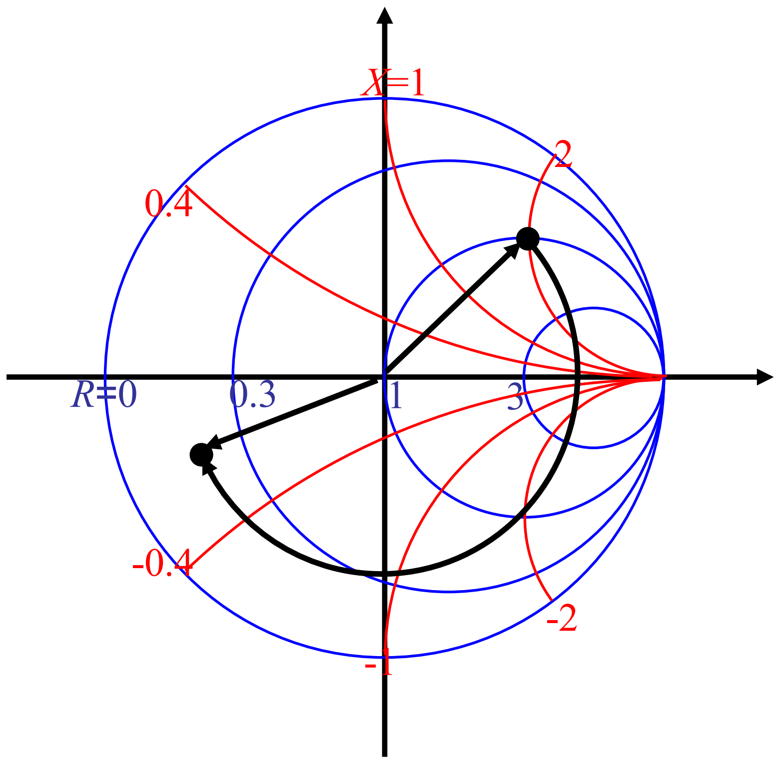
To find out the input impedance

Step 1: find the Z_L point (1+2i)

Step 2: find the Γ_L

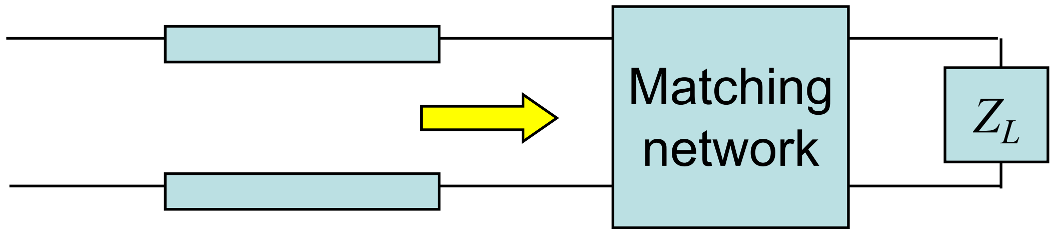
Step 3: moving on the circle (θ=2βz)

Step 4: find the input impedance Z(z)



Impedance matching

$$Z_{in} = Z_0$$

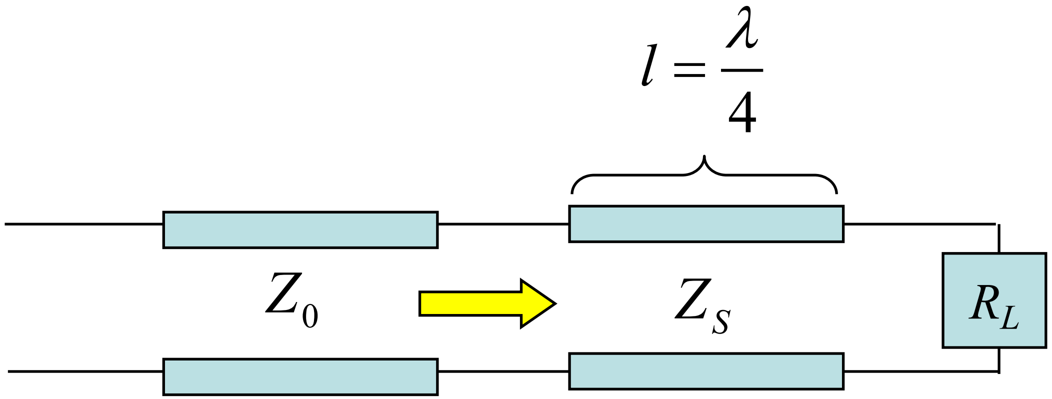


If the matching network consist reactive elements, the power will be transmitted from the source to the load without reflection and loss

Practical matching networks only operate over a narrow bandwidth

The tuning ability of the network for varying load impedances is desired

Quarter wave transformer



When the load is real, one may use a segment of lossless transmission line as the transformer

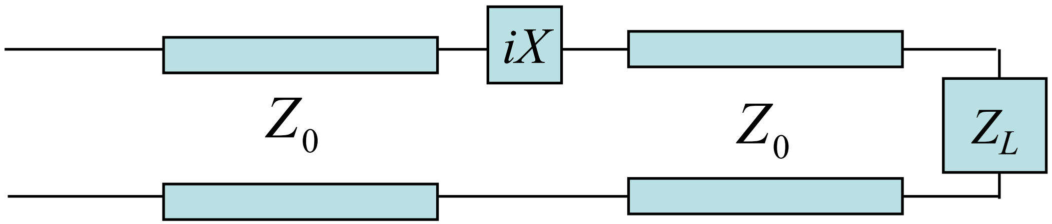
Recall that $$Z_{in} = Z_S \frac{R_L + iZ_S \tan(\beta l)}{Z_S + iR_L \tan(\beta l)}$$

For a quarter-wave-length section $\beta l = \frac{\pi}{2}$ ⇒ $Z_{in} = \frac{Z_S^2}{R_L}$

One may choose $Z_S = \sqrt{R_L Z_0}$ so as to match the impedance

Usage of smith chart

When the load is complex, we may use the following steps to determine the transformer

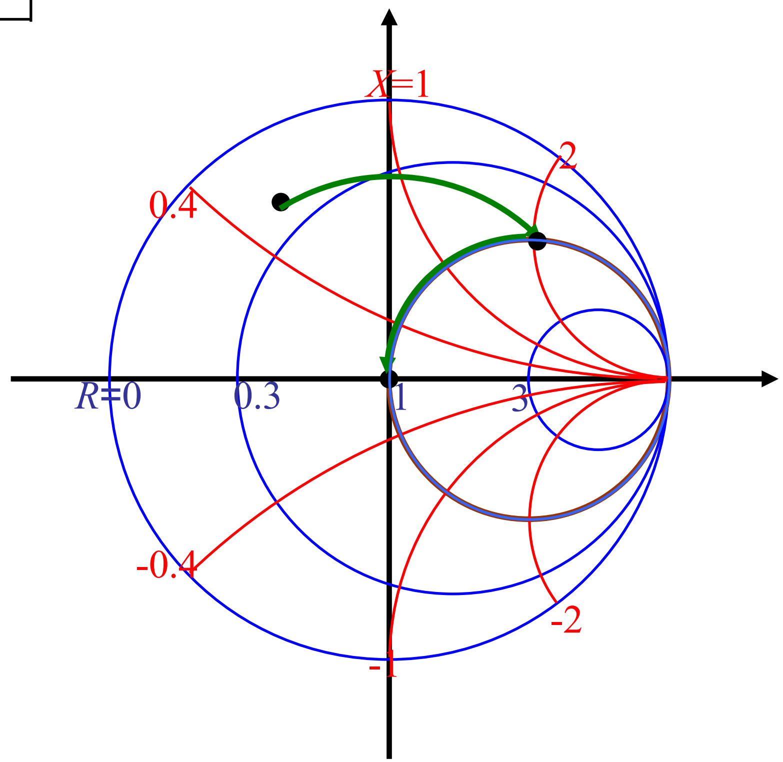


Step 1: find the Z_L point (0.22+0.5i)

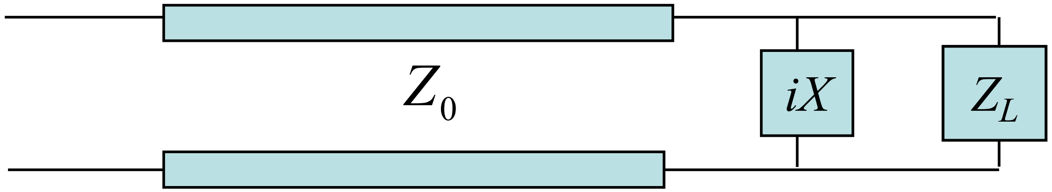
Step 2: find the *target circle* (1+*x*i)

Step 3: move away the load along constant |Γ| circle to the target circle (1+2i)

Step 4: add a reactive element to move to the center iX=-2i



Shunt stubs

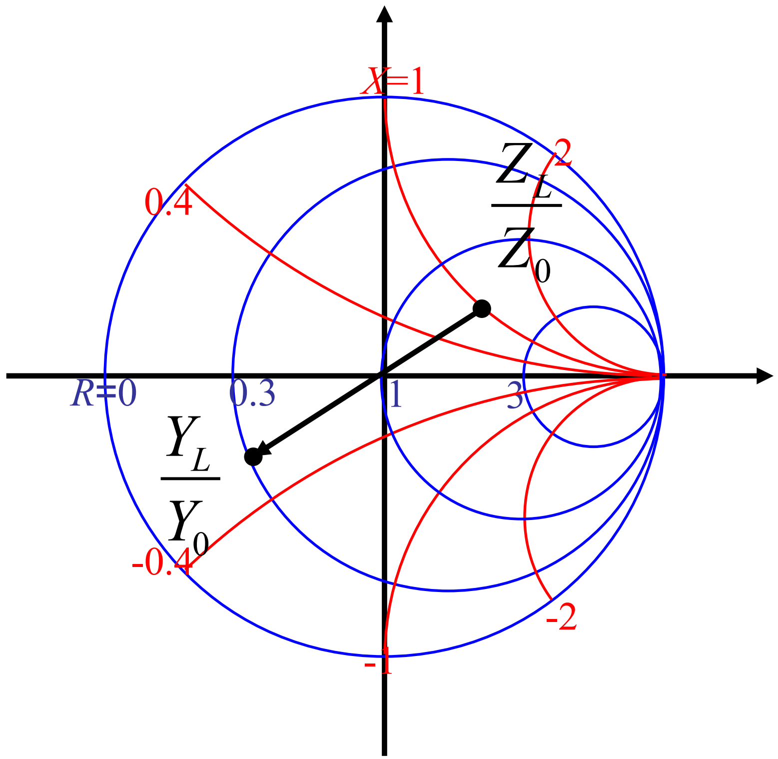


$$Y_{tot} = \frac{1}{iX} + \frac{1}{Z_L}$$

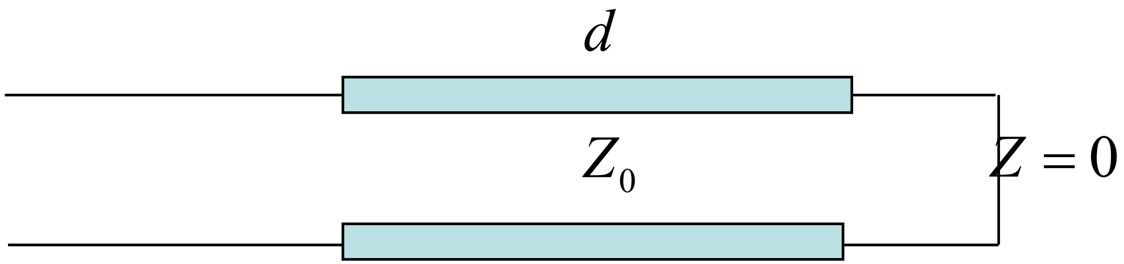
To find the admittance using the smith chart

$$Y_L = \frac{1}{Z_L} \qquad \frac{Y_L}{Y_0} = \frac{Z_0}{Z_L}$$

Simply move the point on the opposite side of the constant $|\Gamma|$ circle



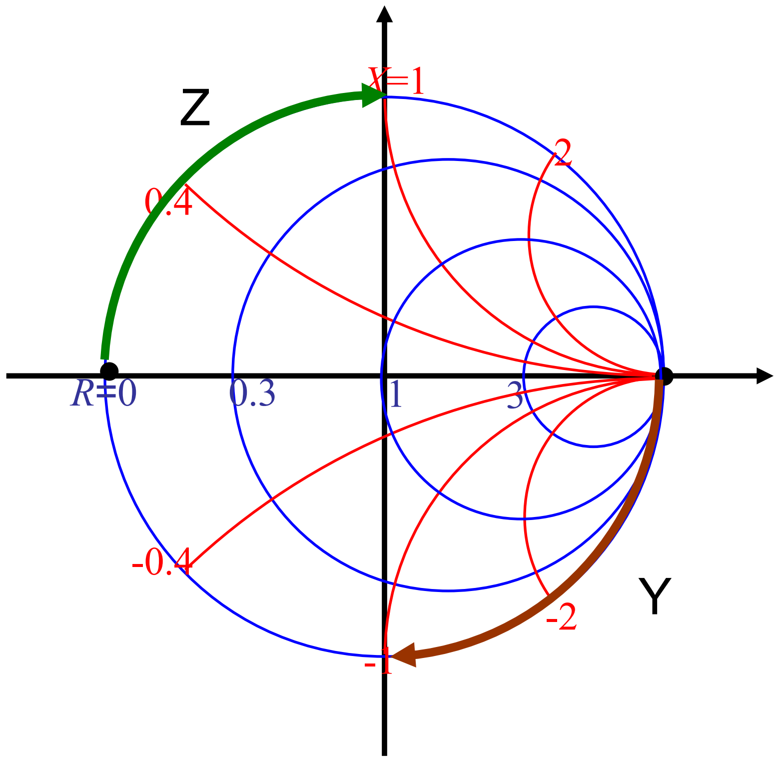
Shorted T-line stub



Step 1: start from the short $Z = 0 \quad Y = \infty$

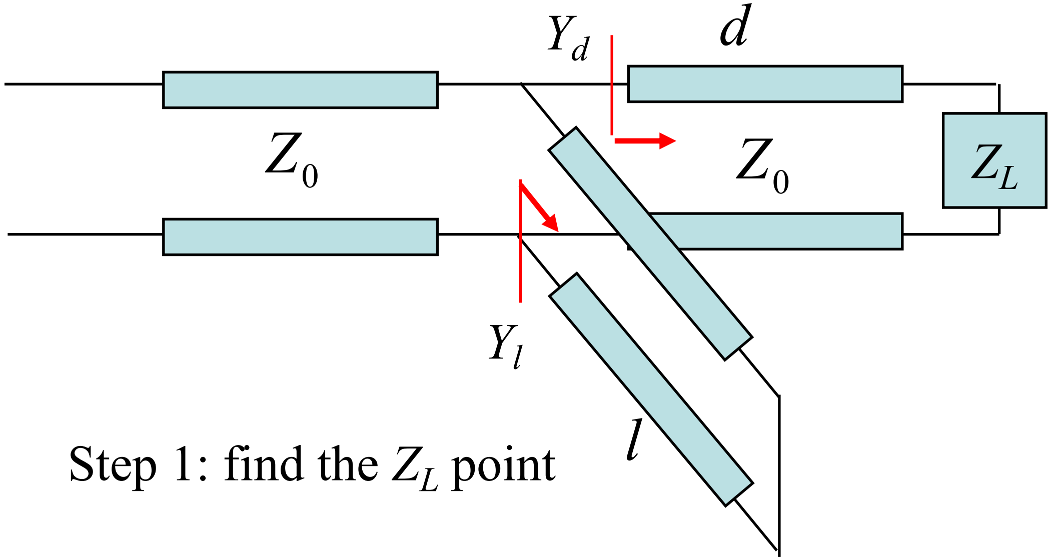
Step 2: move the load along the unit circle

$$Z = iZ_0 \tan(\beta d) \quad Y = -iY_0 \cot(\beta d)$$



The shorted(or open) T-line stubs can be viewed as a reactive tuning elements

Shunt-stub matching



Step 1: find the Z_L point

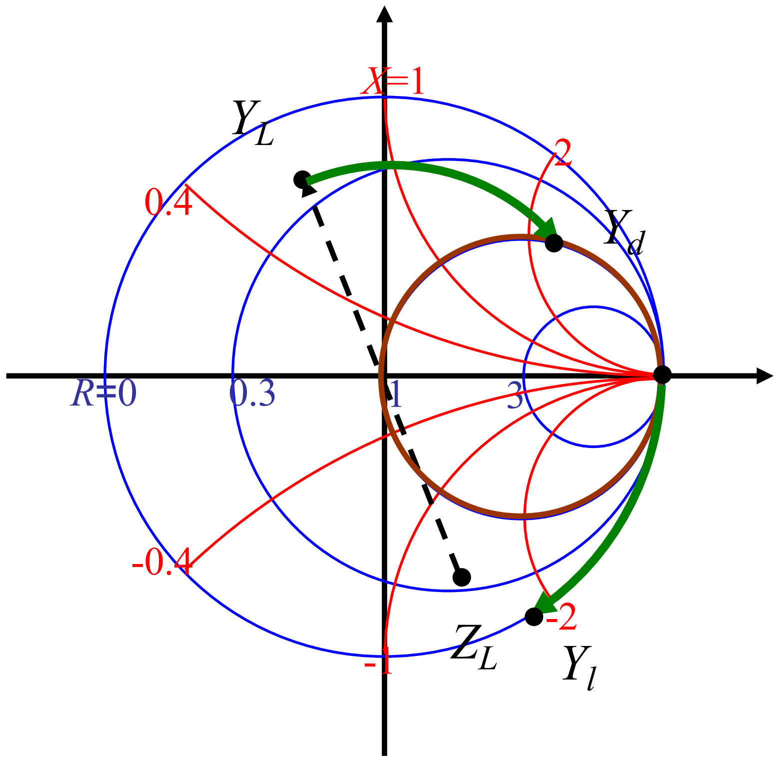
Step 2: find the Y_L point

Step 3: find the 1+ix circle

Step 4: move the point along the constant $|\Gamma|$ circle to the intersection to the 1+ix circle (Y_d=1+ib)

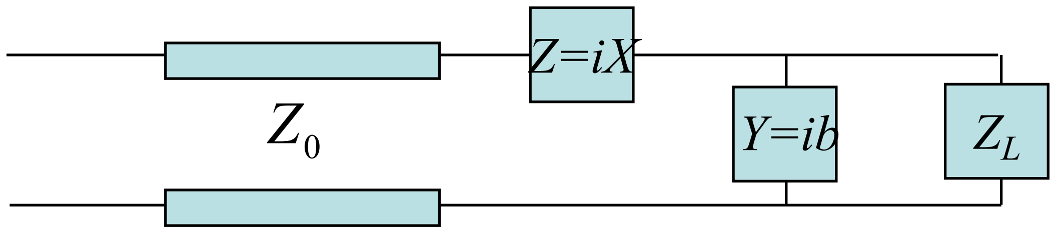
Step 5: find the point representing Y_l=1-ib

Step 6: the total admittance becomes 1+0i



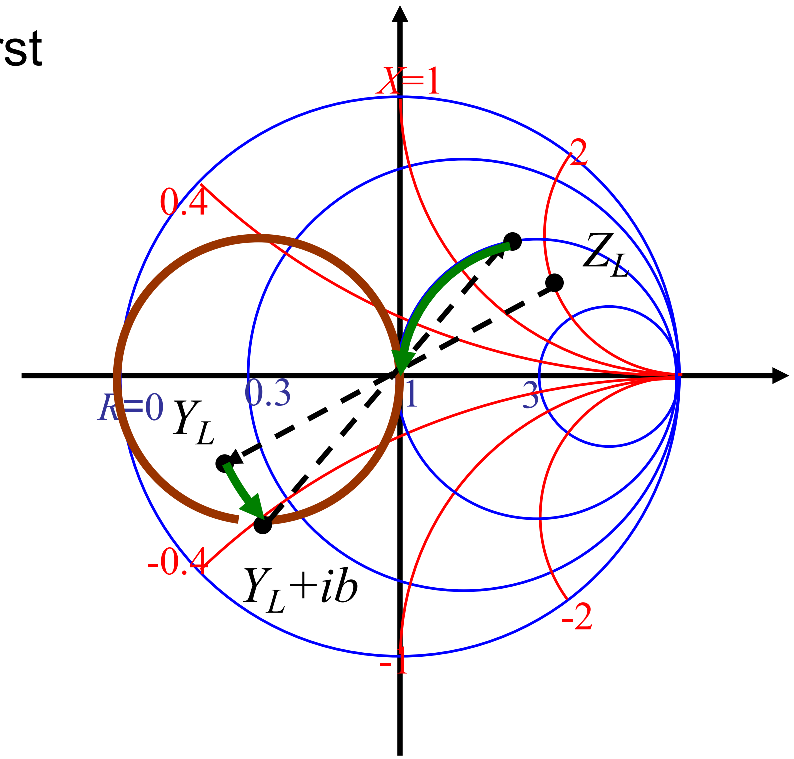
Lumped-element matching (I)

If $|Z_L| > Z_0$



Add a series reactive element will not move to $1+ix$ circle

⇨ Try to add a shunt reactive element first



Step 1: find the Z_L point and Y_L point

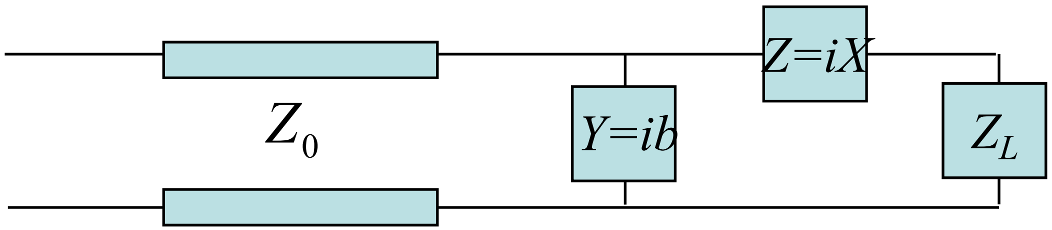
Step 2: move the point along the $1+ix$ circle until it intersect with the *rotated* $1+ix$ circle

Step 3: Now start to add a series element, go back to $Z=(Y_L+ib)^{-1}$

Step 4: move the point along the $1+ix$ circle to the center $Z= (Y_L+ib)^{-1}+iX$

Lumped-element matching (II)

If $|Z_L| < Z_0$



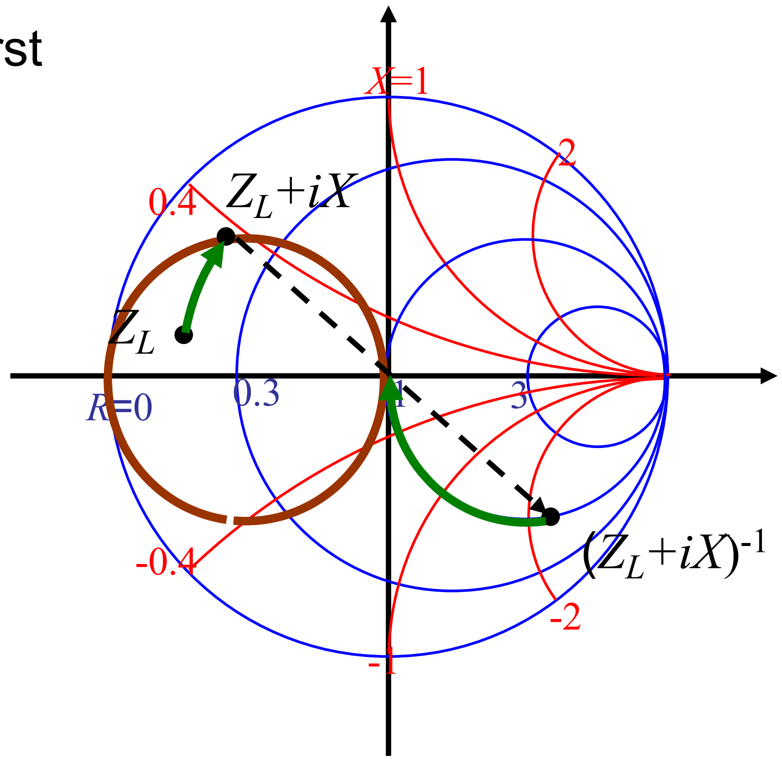
⇨ Try to add a series reactive element first

Step 1: find the Z_L point

Step 2: move the point along the $1+\mathrm{i}x$ circle until it intersect with the *rotated* $1+\mathrm{i}x$ circle (Z_L+iX)

Step 3: Now start to add a series element, go back to $Y=(Z_L+iX)^{-1}$

Step 4: move the point along the $1+\mathrm{i}x$ circle to the center $Y= (Z_L+iX)^{-1}+\mathrm{i}b$



Distributed parameter model

Harmonic waves on TL

Terminated TL

The Smith chart

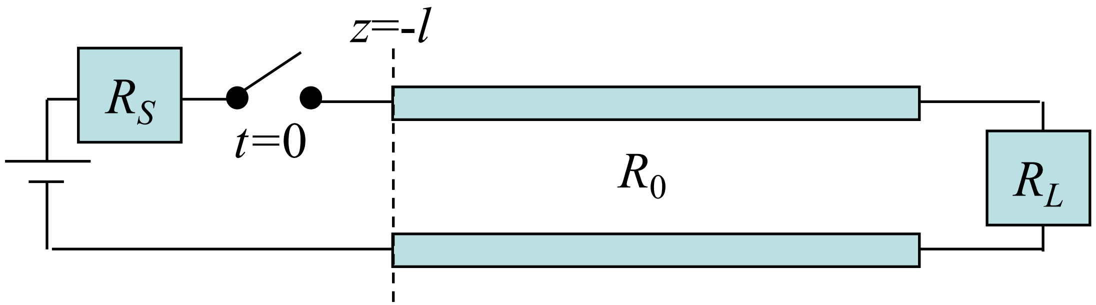
Time domain analysis

S parameters

Transients

When there is a sudden change in voltage or current at one end of a TL

Here we assume a resistive load R_L and resistive source output impedance R_S

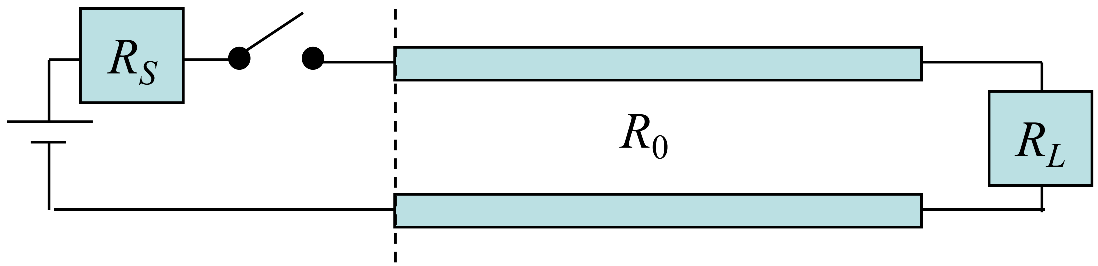


$$\Gamma_S = \frac{R_S - R_0}{R_S + R_0} \qquad \Gamma_L = \frac{R_L - R_0}{R_L + R_0}$$

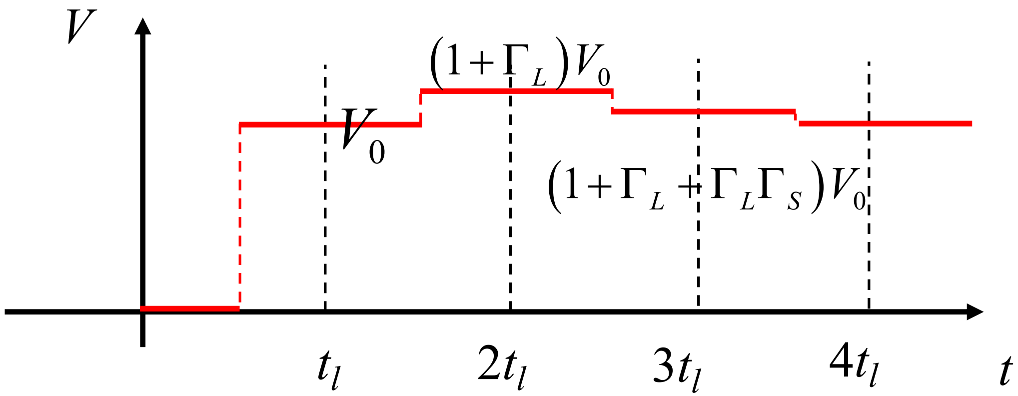
The initial voltage $$V_0 = V_S \frac{R_0}{R_0 + R_S}$$

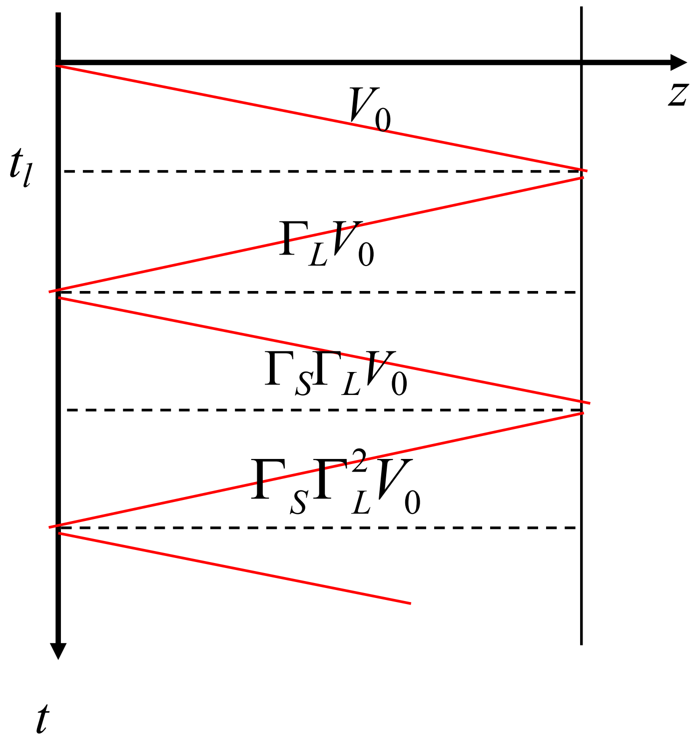
The transit time $$t_l = \frac{l}{v}$$

Bounce diagram

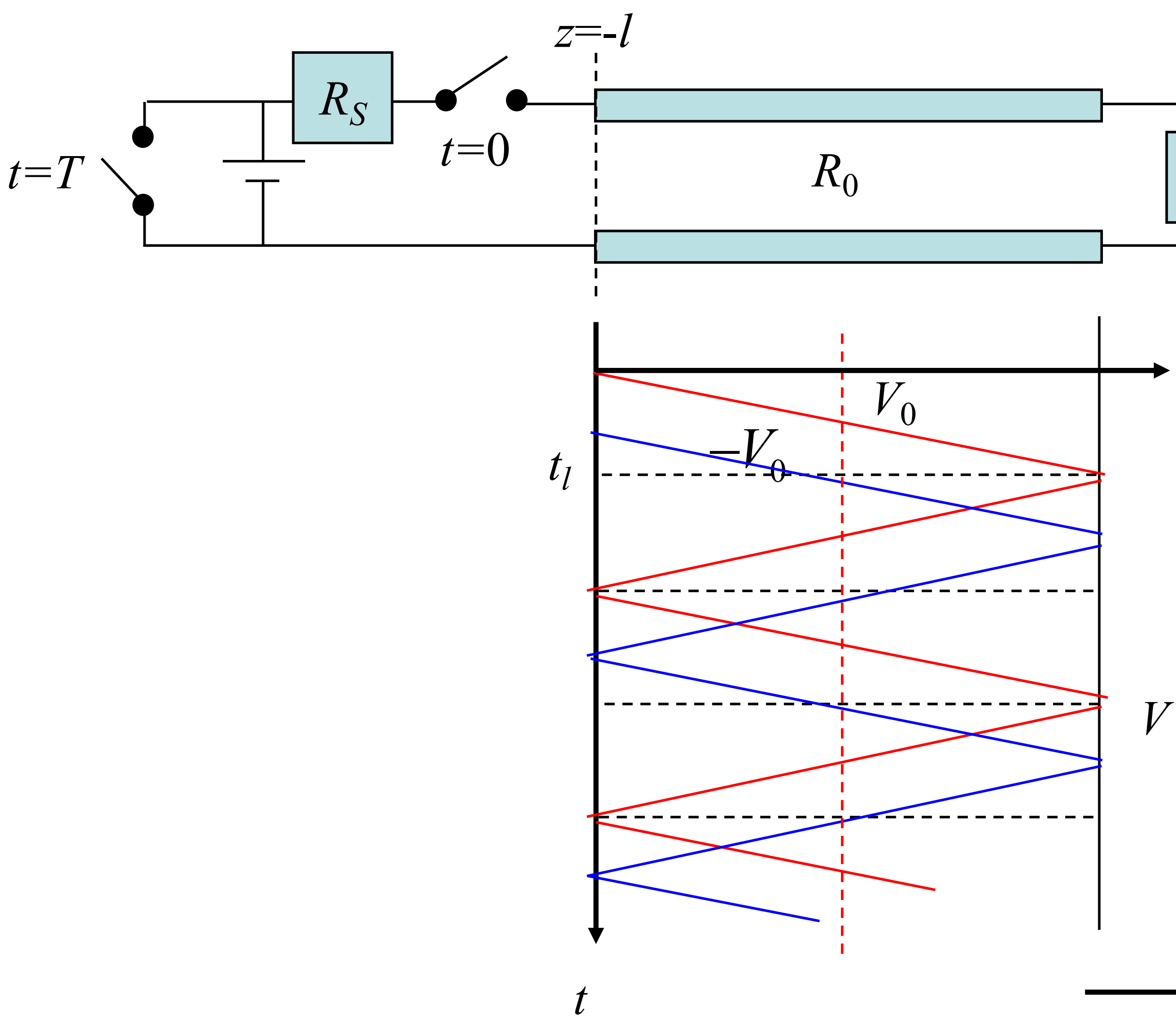


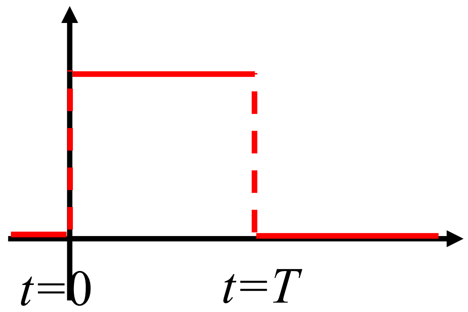
At the middle point





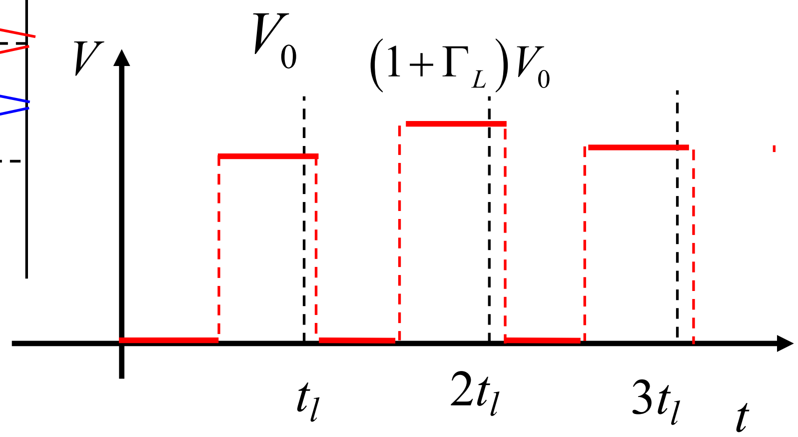
Pulse response





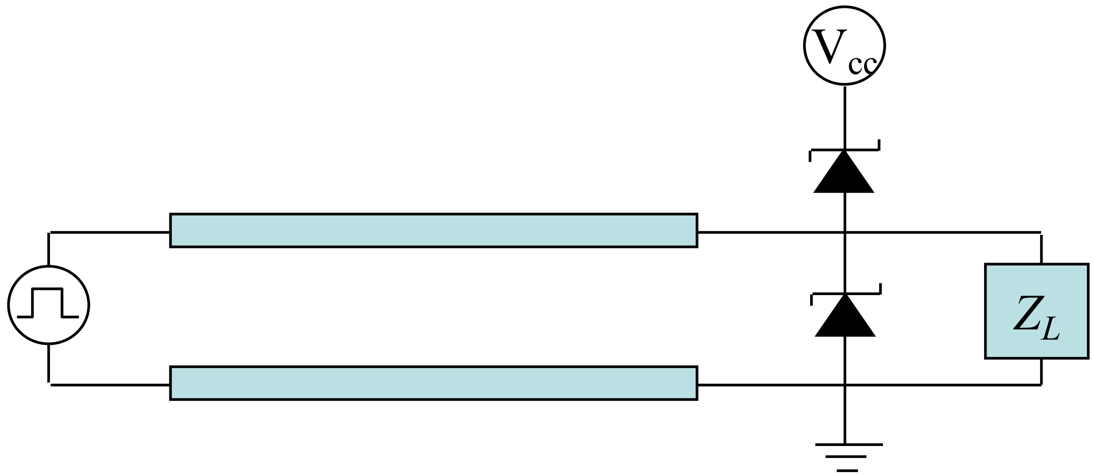
$T < t_l$

At the middle point



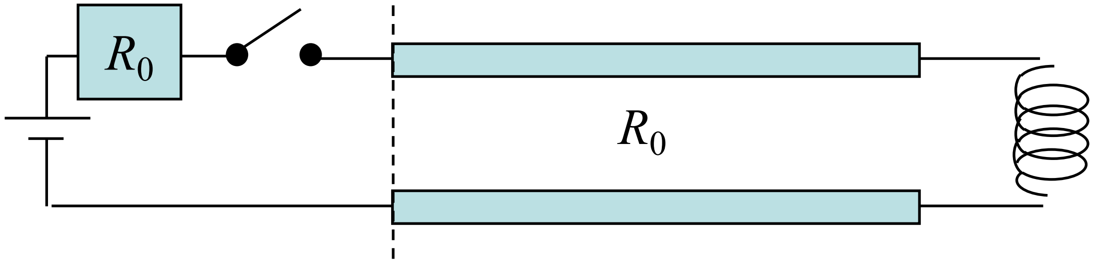
Schottky-diode termination

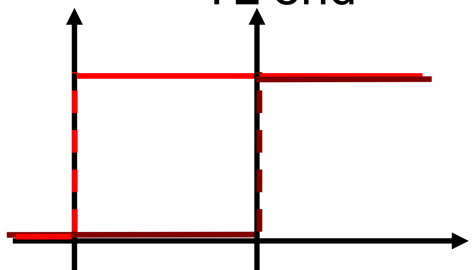
A clamping circuit



The voltage on Z_L will be clamped between V_{cc} and ground

Reactive load-inductor





$$\frac{V_{0,in}}{I_{0,in}} = -\frac{V_{0,re}}{I_{0,re}} = R_0$$

$$I_L = I_{0,in} + I_{0,re} = \frac{1}{R_0}\left(V_{0,in} - V_{0,re}\right)$$

$$V_L = L\frac{dI_L}{dt}$$

$$\Rightarrow \quad V_{0,in} + V_{0,re} = 2V_{0,in} - I_L R_0 = L\frac{dI_L}{dt}$$

$$\Rightarrow \quad L\frac{dI_L}{dt} + I_L R_0 = 2V_{0,in}$$

$$I_L = \frac{2V_{0,in}}{R_0}\left(1 - e^{-R_0 t/L}\right)$$

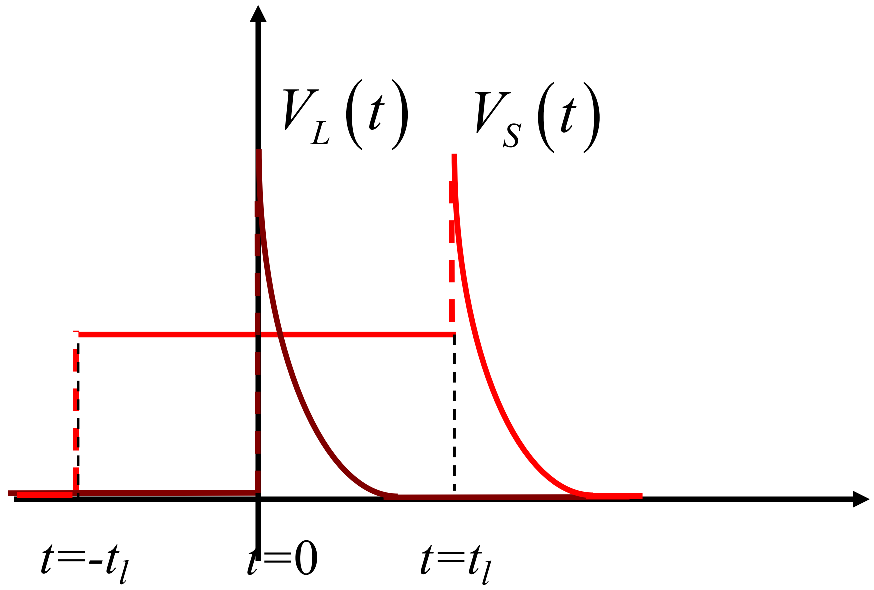
Reactive load-inductor

$$I_L = \frac{2V_{0,in}}{R_0}\left(1 - e^{-R_0 t/L}\right)$$

$$V_L = 2V_{0,in} e^{-R_0 t/L}$$

The reflected voltage is

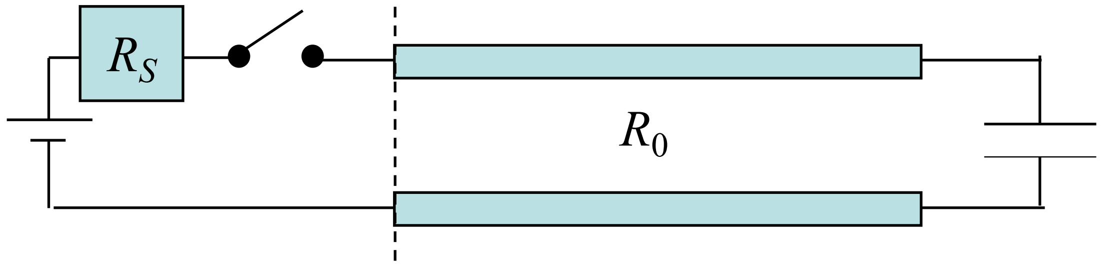
$$V_{0,re} = V_{0,in}\left(2e^{-R_0 t/L} - 1\right)$$



If assume a matched R_S, one have $V_{0,in} = \dfrac{V_{Source}}{2}$

At the source, the voltage is $V_S = V_{0,in} + V_{0,in}\left(2e^{-R_0 \tau/L} - 1\right)\Theta(\tau) \qquad \tau = t - t_l$

Reactive load-capacitor



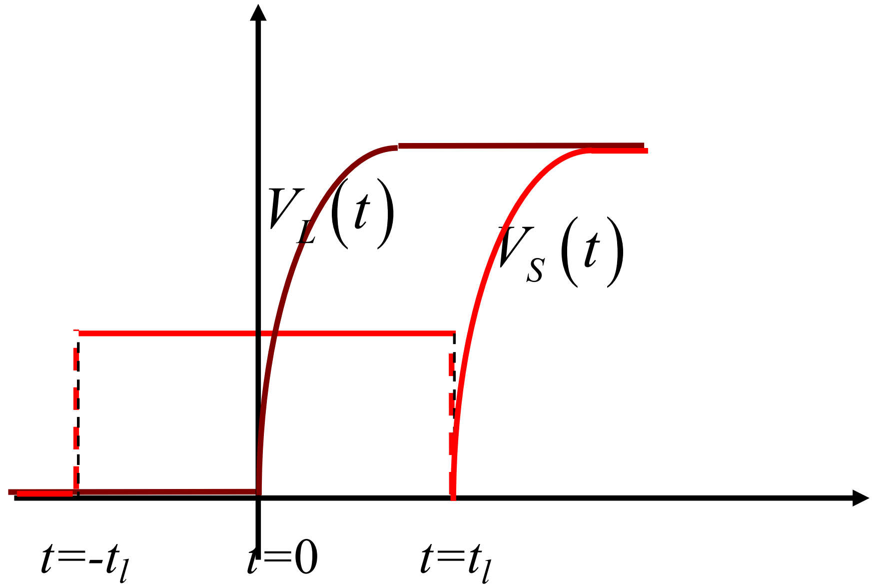
$$I_L = C\frac{dV_L}{dt}$$

$$\Rightarrow \frac{1}{R_0}\left(2V_{0,in} - V_L\right) = C\frac{dV_L}{dt}$$

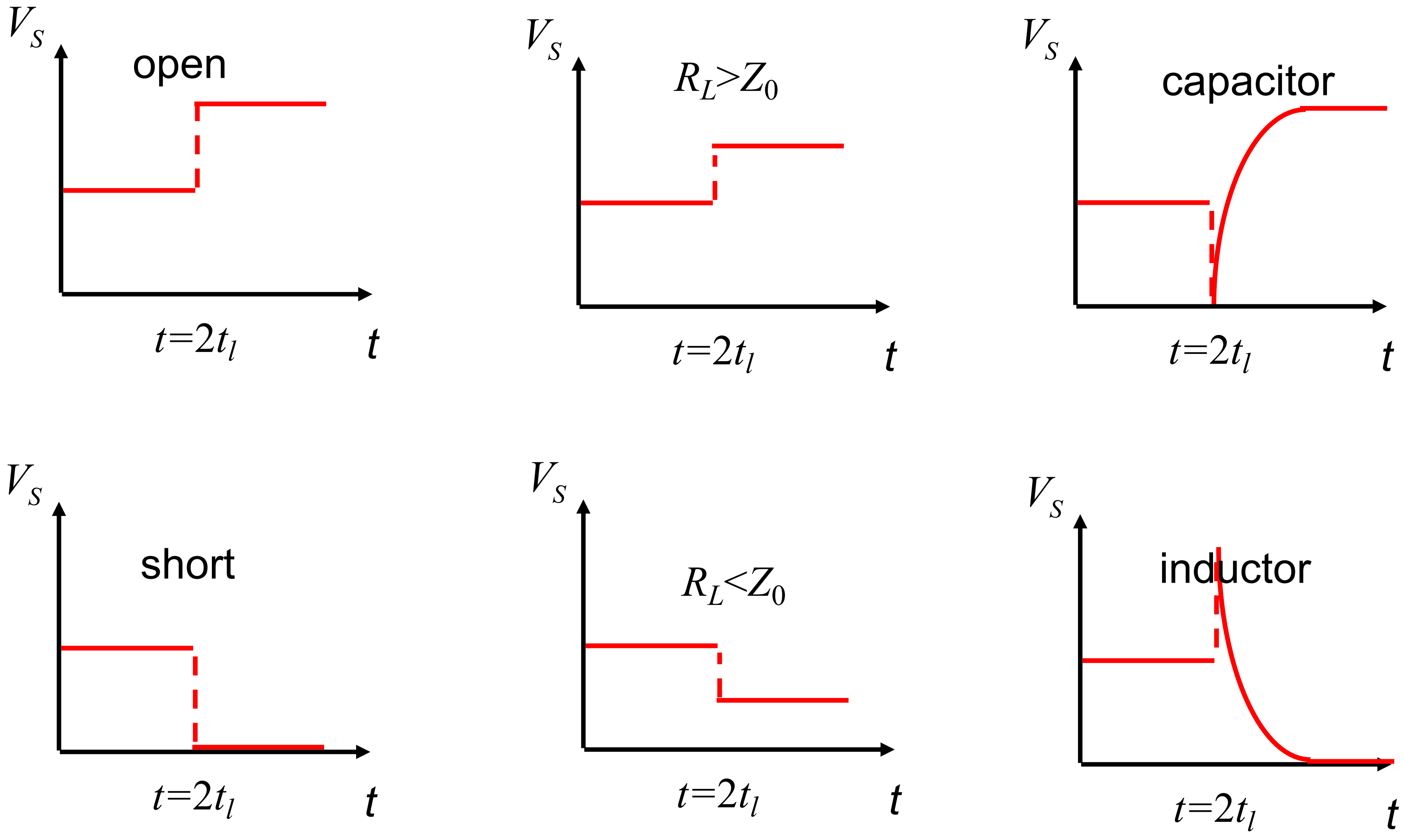
$$R_0 C\frac{dV_L}{dt} + V_L = 2V_{0,in}$$

$$\Rightarrow V_L = 2V_{0,in}\left(1 - e^{-t/R_0 C}\right)$$

$$V_{0,re} = V_{0,in}\left(1 - 2e^{-t/R_0 C}\right)$$



Time domain reflectometry



Dispersion

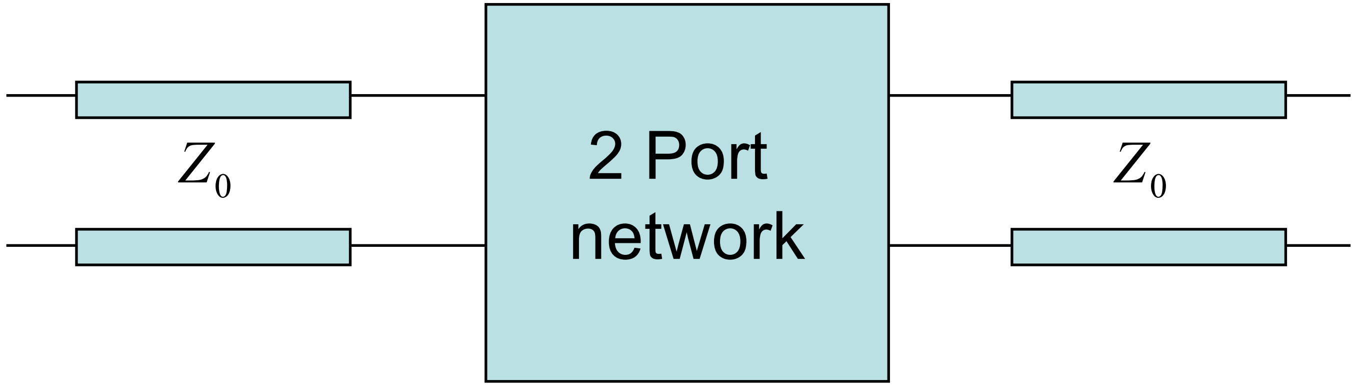
Digital signals are sent as pulses which can be decomposed into sinusoidal components using a Fourier series

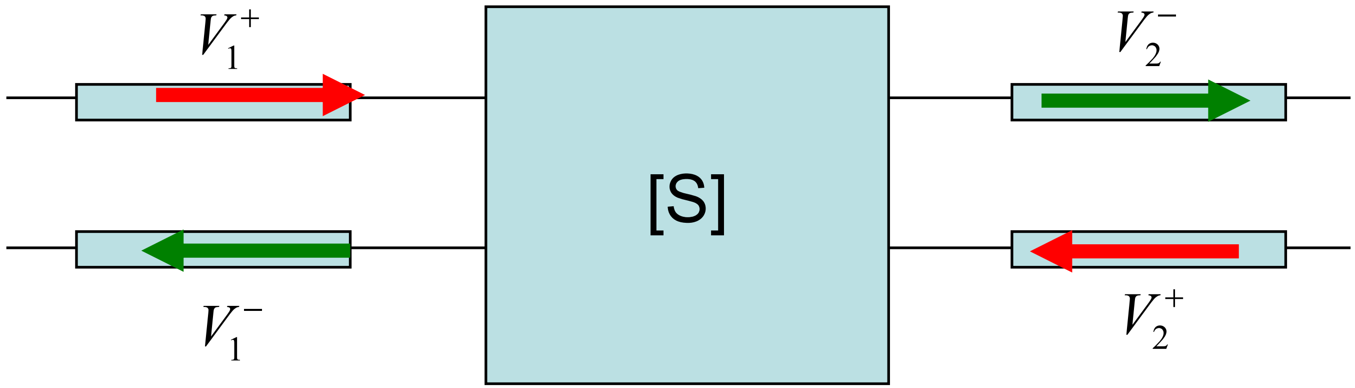
For a realistic medium, the wave traveling speed changes with frequency

The pulse spreads out as it propagates and results in degradation of signal.

Distributed parameter model
Harmonic waves on TL
Terminated TL
The Smith chart
Time domain analysis
S parameters

Scattering parameters

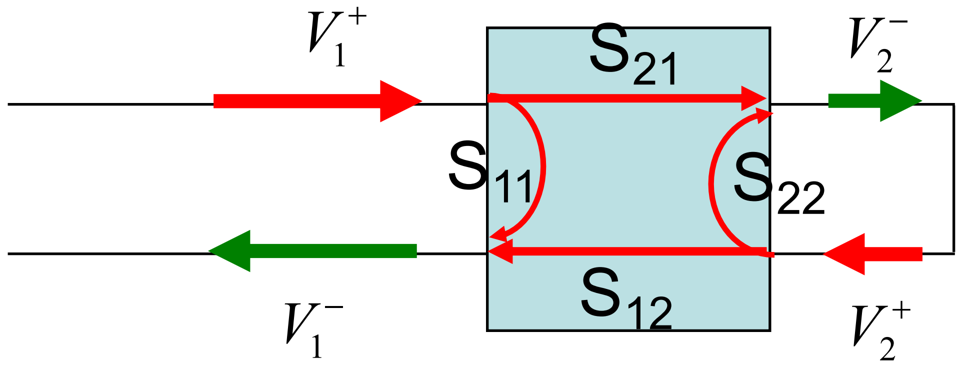




$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

The relation to reflection (I)

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0}$$



If the port 2 is shorted,

$$\Rightarrow \quad V_2^+ + V_2^- = 0$$

$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ = -V_2^+$$

$$V_2^+ = -\frac{S_{21}V_1^+}{1+S_{22}}$$

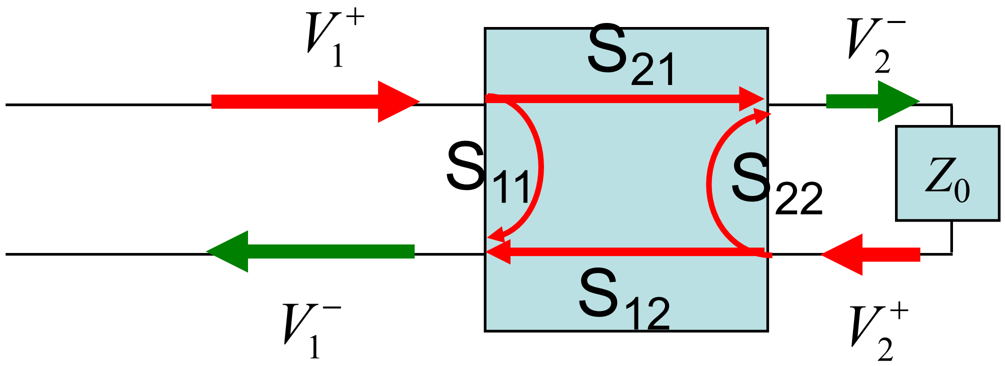
$$\Gamma = \frac{V_1^-}{V_1^+} = S_{11} + S_{12}\frac{V_2^+}{V_1^+} = S_{11} - \frac{S_{12}S_{21}}{1+S_{22}}$$

The relation to reflection (II)

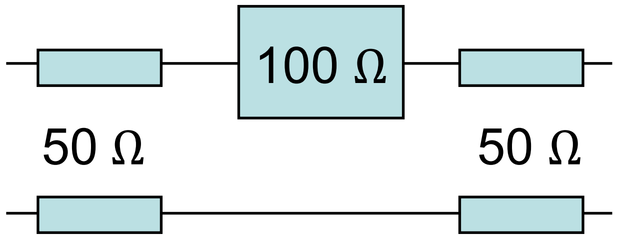
If the port 2 is connected to a Z_0 load

$$V_2^+ = 0$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \Gamma$$

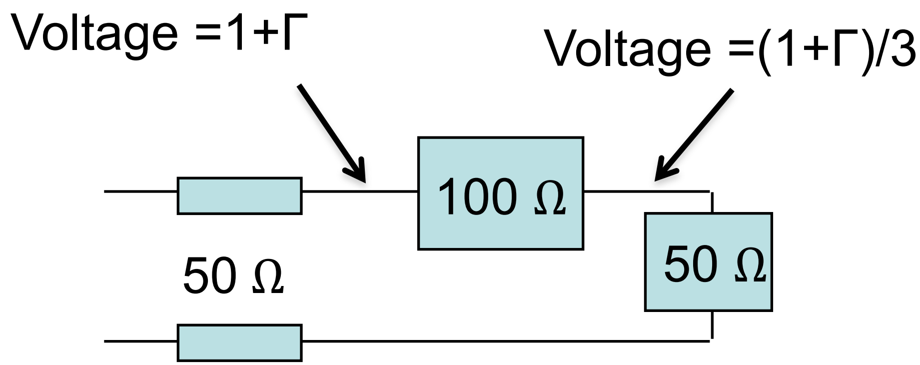


Suppose the network is



$$S_{11} = \Gamma = \frac{150\Omega - 50\Omega}{150\Omega + 50\Omega} = \frac{1}{2}$$

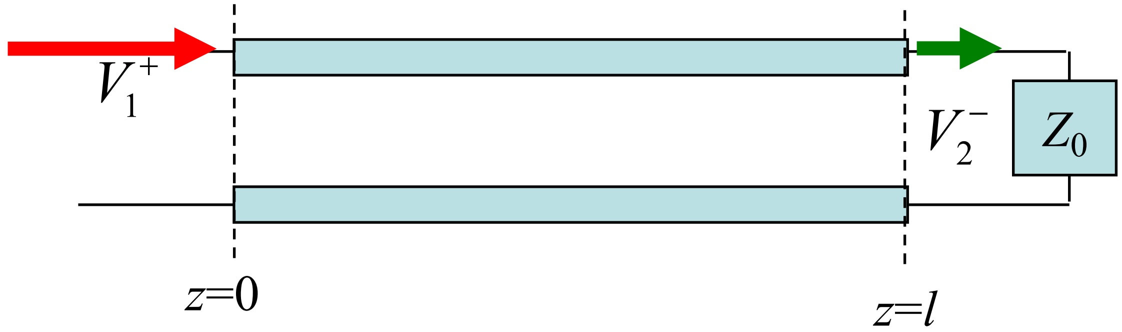
$$S_{21} = \frac{1+\Gamma}{3} = \frac{1}{2}$$



$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Shift of reference plane

The S-matrix of a section of TL:

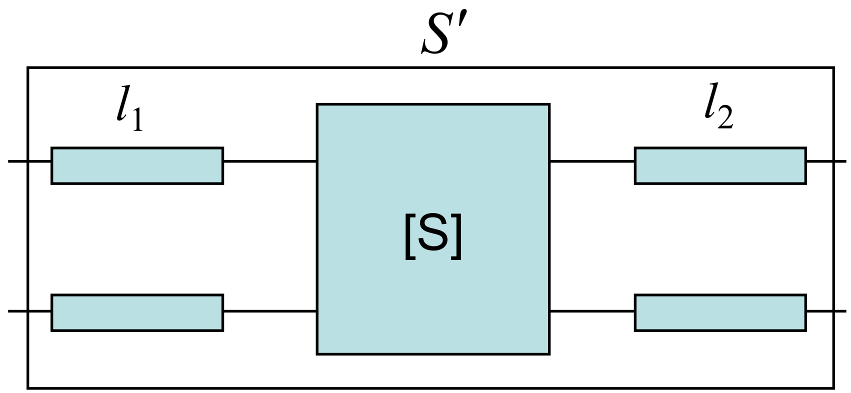


If the TL is terminated by a matched load

$$S_{21}=\frac{V_2^-}{V_1^+}=e^{-i\beta l}$$

$$V_2^- = V_1^+ e^{-i\beta l}$$

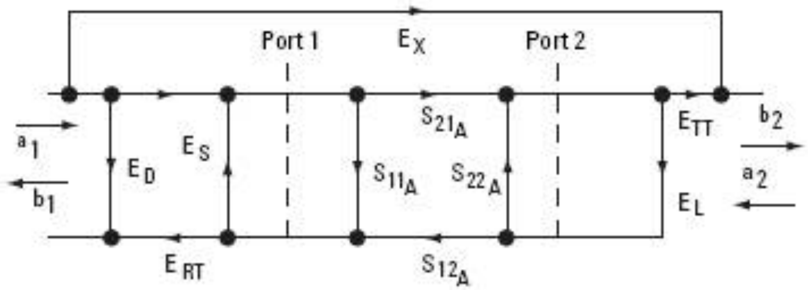
$$S=\begin{bmatrix} 0 & e^{-i\beta l} \\ e^{-i\beta l} & 0 \end{bmatrix}$$



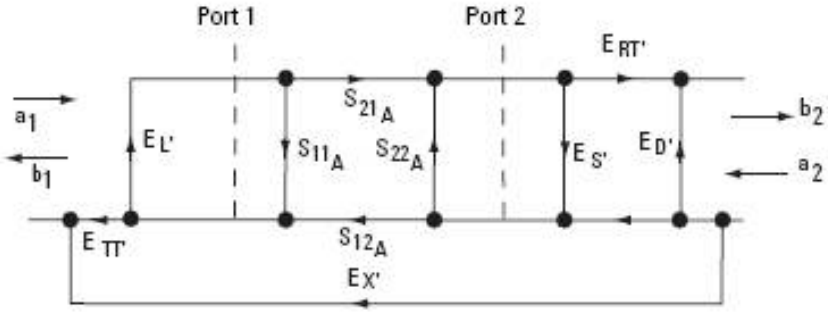
$$\Rightarrow \quad S'=\begin{bmatrix} S_{11}e^{-2i\beta l_1} & S_{12}e^{-i\beta(l_1+l_2)} \\ S_{21}e^{-i\beta(l_1+l_2)} & S_{22}e^{-2i\beta l_2} \end{bmatrix}$$

12-parameter calibration

Forward model



Reverse model



E_D = Fwd Directivity

E_S = Fwd Source Match

E_{RT} = Fwd Reflection Tracking

E_L = Fwd Load Match

E_{TT} = Fwd Transmission Tracking

E_X = Fwd Isolation

$E_{D'}$ = Rev Directivity

$E_{S'}$ = Rev Source Match

$E_{RT'}$ = Rev Reflection Tracking

$E_{L'}$ = Rev Load Match

$E_{TT'}$ = Rev Transmission Tracking

$E_{X'}$ = Rev Isolation

$$S_{11a} = \frac{\left(\frac{S_{11m}-E_D}{E_{RT}}\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}E_S'\right) - E_L\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}{\left(1+\frac{S_{11m}-E_D}{E_{RT}}E_S\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}E_S'\right) - E_L'E_L\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}$$

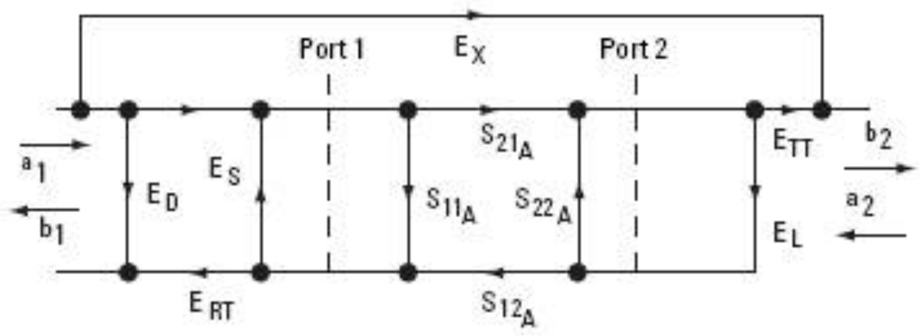
$$S_{21a} = \frac{\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}(E_S'-E_L)\right)}{\left(1+\frac{S_{11m}-E_D}{E_{RT}}E_S\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}E_S'\right) - E_L'E_L\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}$$

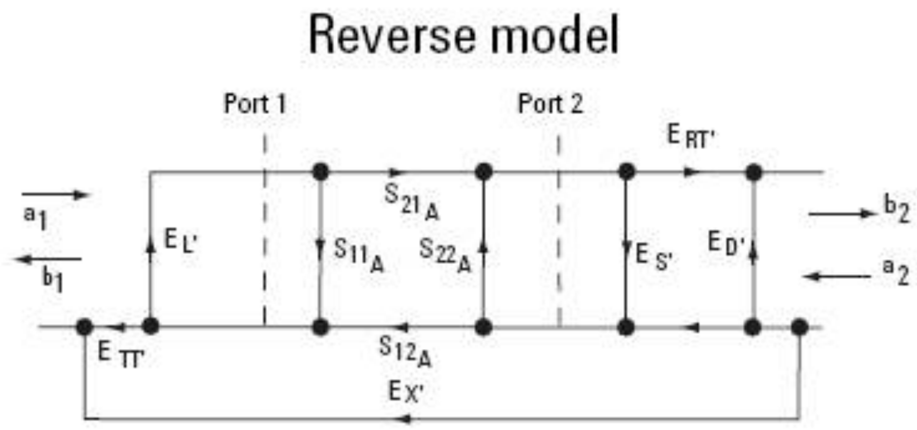
$$S_{12a} = \frac{\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)\left(1+\frac{S_{11m}-E_D}{E_{RT}}(E_S-E_L')\right)}{\left(1+\frac{S_{11m}-E_D}{E_{RT}}E_S\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}E_S'\right) - E_L'E_L\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}$$

$$S_{22a} = \frac{\left(\frac{S_{22m}-E_D'}{E_{RT}'}\right)\left(1+\frac{S_{11m}-E_D}{E_{RT}}E_S\right) - E_L'\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}{\left(1+\frac{S_{11m}-E_D}{E_{RT}}E_S\right)\left(1+\frac{S_{22m}-E_D'}{E_{RT}'}E_S'\right) - E_L'E_L\left(\frac{S_{21m}-E_X}{E_{TT}}\right)\left(\frac{S_{12m}-E_X'}{E_{TT}'}\right)}$$

- Notice that each actual S-parameter is a function of all four measured S-parameters
- Analyzer must make forward and reverse sweep to update any one S-parameter

The way to determine parameters





Connect load at port 1

$$S_{11}(L1) = E_D$$
$$S_{21}(L1) = E_X$$

Connect load at port 2

$$S_{12}(L2) = E_{X'}$$
$$S_{22}(L2) = E_{D'}$$

Now we redefine the new measurement S-parameters as

$$S_{11} = S_{11} - E_D$$
$$S_{21} = S_{21} - E_X$$
$$S_{12} = S_{12} - E_{X'}$$
$$S_{22} = S_{22} - E_{D'}$$

Port 1 open and short

Connect open at port 1

$$S_{11}^{0}=1$$

⇨

$$1+E_S\frac{S_{11}(O1)}{E_{RT}}=\frac{S_{11}(O1)}{E_{RT}}$$

Connect short at port 1

$$S_{11}^{0}=-1$$

$$1+E_S\frac{S_{11}(S1)}{E_{RT}}=-\frac{S_{11}(S1)}{E_{RT}}$$

$$S_{21}(O1)=S_{21}(S1)=E_X$$

⇨

$$(1-E_S)S_{11}(O1)=E_{RT}$$

$$(1+E_S)S_{11}(S1)=-E_{RT}$$

$$E_S=\frac{S_{11}(O1)-S_{11}(S1)}{S_{11}(S1)+S_{11}(O1)}$$

$$E_{RT}=\frac{2S_{11}(O1)S_{11}(S1)}{S_{11}(S1)-S_{11}(O1)}$$

Port 2 open and short

Connect open at port 2

$$S_{22}^{0} = 1$$

$$1 + \frac{E_S' S_{22}(O2)}{E_{RT'}} = \frac{S_{22}(O2)}{E_{RT'}}$$

Connect short at port 2

$$S_{22}^{0} = -1$$

$$1 + \frac{E_S' S_{22}(S2)}{E_{RT}'} = -\frac{S_{22}(S2)}{E_{RT}'}$$

$$S_{12}(O2) = S_{12}(S2) = E_{X'}$$

$$\begin{cases} E_S' = \dfrac{S_{22}(O2) - S_{22}(S2)}{S_{22}(S2) + S_{22}(O2)} \\[2ex] E_{RT}' = \dfrac{2S_{22}(O2)\,S_{22}(S2)}{S_{22}(S2) - S_{22}(O2)} \end{cases}$$

Through

Connect through at port 1 and port 2

$$S_{12}^0 = S_{21}^0 = 1$$

$$S_{11}^0 = S_{22}^0 = 0$$

$$1 + E_S \frac{S_{11}(T)}{E_{RT}} = \frac{S_{21}(T)}{E_{TT}} \qquad 1 + E_S{}' \frac{S_{22}(T)}{E_{RT}{}'} = \frac{S_{12}(T)}{E_{TT}{}'}$$

$$E_L \frac{S_{21}(T)}{E_{TT}} = \frac{S_{11}(T)}{E_{RT}} \qquad E_L{}' \frac{S_{12}(T)}{E_{TT}{}'} = \frac{S_{22}(T)}{E_{RT}{}'}$$

$$\Rightarrow \quad E_{TT} = \frac{S_{21}(T)}{1 + E_S \dfrac{S_{11}(T)}{E_{RT}}} \qquad E_{TT}{}' = \frac{S_{12}(T)}{1 + E_S{}' \dfrac{S_{22}(T)}{E_{RT}{}'}}$$

Through

Once $E_{RT}, E_{TT}, E'_{RT}, E'_{TT}$ are determined, one may redefine

$$\frac{S_{11}}{E_{RT}} \to \tilde{S}_{11}$$

$$\frac{S_{21}}{E_{TT}} \to \tilde{S}_{21}$$

$$\frac{S_{12}}{E'_{TT}} \to \tilde{S}_{12}$$

$$\frac{S_{22}}{E'_{RT}} \to \tilde{S}_{22}$$

$$E_L = \frac{\tilde{S}_{11}(T)}{\tilde{S}_{21}(T)}$$

$$E'_L = \frac{\tilde{S}_{22}(T)}{\tilde{S}_{12}(T)}$$

Then all the 12 parameters are determined

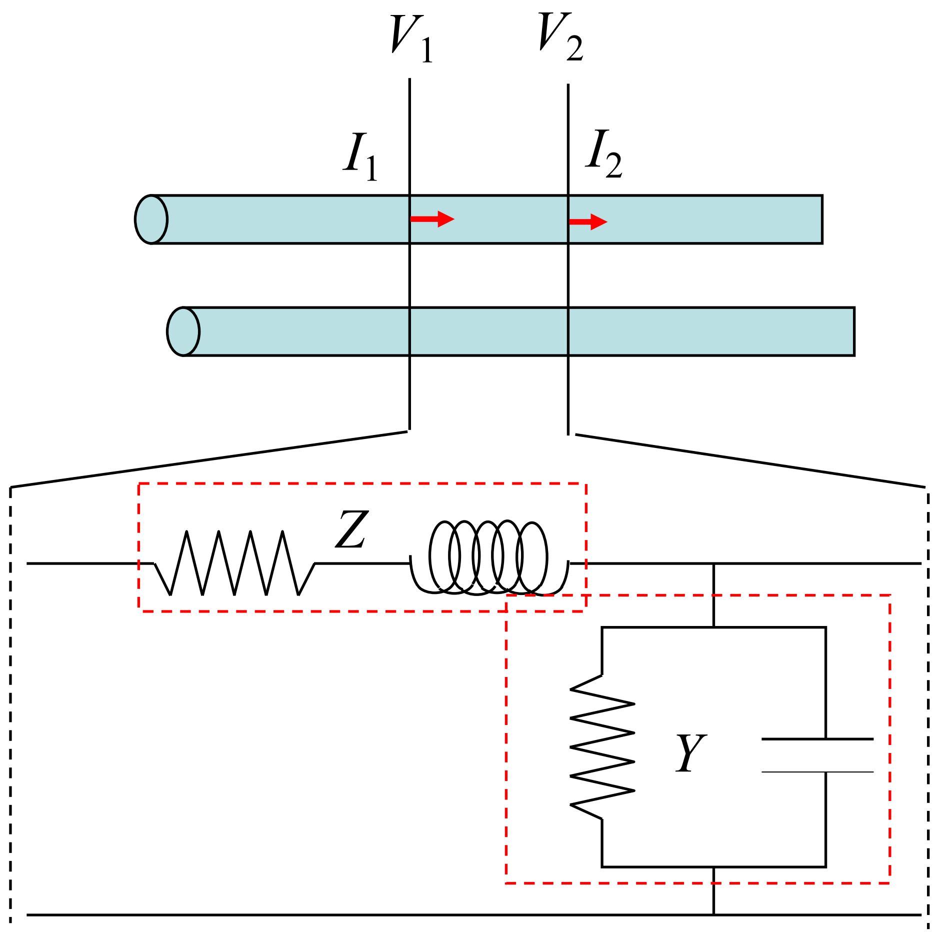
ABCD matrix

Also called ray transfer matrix

It is a type of ray tracing technique used in the design of some optical systems

In microwave engineering, we may define

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$



ABCD matrix for elements

For Y=0

$$V_1 = V_2 + ZI_2$$

$$I_1 = I_2$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$$

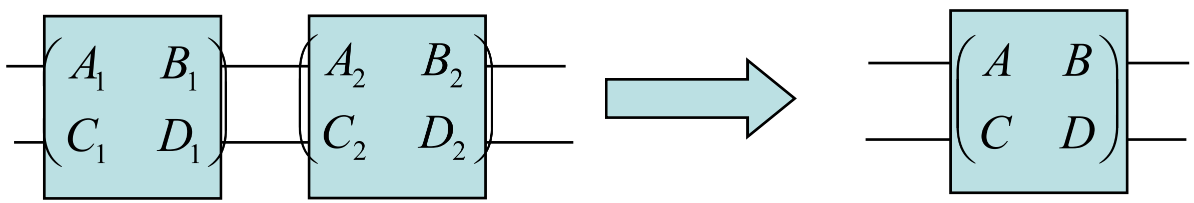
For Z=0

$$V_1 = V_2$$

$$I_1 = I_2 + GV_2$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ G & 1 \end{pmatrix}$$

Two elements connected in series



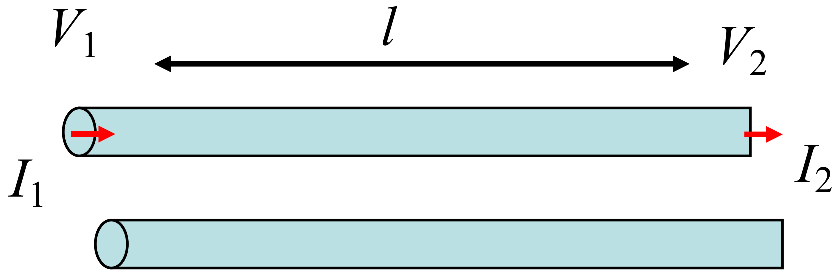
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

ABCD matrix for transmission lines

The solutions to the transmission lines

$$V(z) = V_+ e^{-\gamma z} + V_- e^{\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left(V_+ e^{-\gamma z} - V_- e^{\gamma z}\right)$$



$$V_2 = V(0) = V_+ + V_-$$

$$I_2 = I(0) = \frac{1}{Z_0}\left(V_+ - V_-\right)$$

$$V_+ = \frac{V_2 + Z_0 I_2}{2}$$

$$V_- = \frac{V_2 - Z_0 I_2}{2}$$

ABCD matrix for transmission lines

$$V_1 = V(-l) = V_+ e^{\gamma l} + V_- e^{-\gamma l}$$

$$= \frac{e^{\gamma l} + e^{-\gamma l}}{2} V_2 + \frac{e^{\gamma l} - e^{-\gamma l}}{2} Z_0 I_2$$

$$= \cosh(\gamma l) V_2 + \sinh(\gamma l) Z_0 I_2$$

$$I_1 = I(-l) = \frac{V_+}{Z_0} e^{\gamma l} - \frac{V_-}{Z_0} e^{-\gamma l}$$

$$= \frac{e^{\gamma l} - e^{-\gamma l}}{2Z_0} V_2 + \frac{e^{\gamma l} + e^{-\gamma l}}{2} I_2$$

$$= \frac{1}{Z_0} \cosh(\gamma l) V_2 + \sinh(\gamma l) I_2$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cosh(\gamma l) & \sinh(\gamma l) Z_0 \\ \frac{1}{Z_0} \sinh(\gamma l) & \cosh(\gamma l) \end{pmatrix}$$

Homework II

Ch6
42, 47, 56, 59

Ch10
13 15 16 21