Simple harmonic oscillations

Equation of motion

- Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

- The equations of motion in the Heisenberg picture are

$$i\hbar\dot{p} = [p, H] = -i\hbar m\omega^2 q$$

$$i\hbar\dot{q} = [q, H] = i\hbar\frac{p}{m}$$

Uncertainty principle

- Assume that low-lying states have coordinate and momentum uncertainties

$$\Delta p \Delta q \sim \hbar$$

$$E \sim \frac{\Delta p^2}{2m} + \frac{1}{2} m\omega^2 \Delta q^2 \sim \frac{\hbar^2}{2m\Delta q^2} + \frac{1}{2} m\omega^2 \Delta q^2$$

- this has its minimum at

$$E_{\text{min}} \sim \hbar\omega$$

$$\Delta q^2 \sim \frac{\hbar}{m\omega}$$

Dimensionless variables

- redefine the Hamiltonian, the canonical variables and time in terms of the following dimensionless quantities:

$$\frac{H}{\hbar\omega} \longrightarrow H \qquad \omega t \longrightarrow t$$

$$\frac{q}{\sqrt{\frac{\hbar}{m\omega}}} \longrightarrow q \qquad \frac{p}{\sqrt{m\hbar\omega}} \longrightarrow p$$

- With new variables, we have

$$H = \frac{p^2}{2} + \frac{q^2}{2} \qquad [q, p] = i$$

a and *a*⁺

- introducing the following non-Hermitian operators:

$$a = \frac{1}{\sqrt{2}}\left(q + ip\right) \qquad a^{\dagger} = \frac{1}{\sqrt{2}}\left(q - ip\right)$$

$$\left[a, a^{\dagger}\right] = 1$$

- The Hamiltonian now reads

$$H = \frac{1}{2}\left(p^2 + q^2\right) = \frac{1}{4}\left(a + a^{\dagger}\right)^2 - \frac{1}{4}\left(a - a^{\dagger}\right)^2 = \frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right) = a^{\dagger}a + \frac{1}{2}$$

Time evolution of *a* and *a*⁺

- Heisenberg equation of motion for any operator (dimensionless operator)

$$i\dot{Q} = \left[Q, H\right]$$

- Equations of motions (decoupled equations)

$$i\dot{a} = [a, H] = a$$

$$i\dot{a}^\dagger = \left[a^\dagger, H\right] = -a^\dagger$$

- Time dependence

$$a(t) = e^{-it}a(0) \qquad\qquad a^\dagger(t) = e^{it}a^\dagger(0)$$

Time evolution of p and q

$$q(t) = \frac{1}{\sqrt{2}}\left[a(t) + a^{\dagger}(t)\right] = q(0)\cos t + p(0)\sin(t)$$

$$p(t) = \frac{1}{\sqrt{2}i}\left[a(t) - a^{\dagger}(t)\right] = p(0)\cos t - q(0)\sin(t)$$

$$\left[q(t), q(0)\right] = \left[p(0), q(0)\right]\sin t = -i\sin t$$

- with dimensional variables

$$\left[q(t), q(0)\right] = -i\frac{\hbar}{m\omega}\sin\omega t$$

Number operator

- Define $$N = a^\dagger a \qquad\qquad H = N + \frac{1}{2}$$
- N should be Hermitian and positive definite
$$\langle N \rangle \geq 0$$
- Let |n> be the eigenstate of N (and H)
$$N|n\rangle = n|n\rangle$$
- We want to study the properties of state
$$|\phi\rangle = a|n\rangle \qquad\qquad |\phi'\rangle = a^\dagger|n\rangle$$

$$[N, a] = \left[a^\dagger a, a\right] = -\,a$$

$$N|\phi\rangle = Na|n\rangle = aN|n\rangle + [N, a]|n\rangle = (n-1)a|n\rangle = (n-1)|\phi\rangle$$

$$\Longrightarrow \qquad |\phi\rangle \propto |n-1\rangle$$

$$\left[N, a^\dagger\right] = \left[a^\dagger a, a^\dagger\right] = a^\dagger$$

$$N|\phi'\rangle = Na^\dagger|n\rangle = a^\dagger N|n\rangle + \left[N, a^\dagger\right]|n\rangle = (n+1)a^\dagger|n\rangle = (n+1)|\phi'\rangle$$

$$\Longrightarrow \qquad |\phi'\rangle \propto |n+1\rangle$$

To determine the normalization constant

$$\langle\phi|\phi\rangle = \langle n|a^\dagger a|n\rangle = \langle n|N|n\rangle = n$$

$$\langle\phi'|\phi'\rangle = \langle n|aa^\dagger|n\rangle = n+1$$

Energy Eigenvalues

- One may choose the arbitrary phase to be zero

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad\qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

- *n* should be integers, otherwise *N* would not be positive

- The energy spectrum (dimensional variable)

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

- a^+ and a are respectively called creation and destruction operators, presenting the addition or removal of an excitation

- matrix elements of a^+ and a

$$\langle n | a^\dagger | n' \rangle = \sqrt{n}\delta_{n,n'+1} \qquad \langle n | a | n' \rangle = \sqrt{n+1}\delta_{n,n'-1}$$

- for the canonical variables

$$\langle n | q | n' \rangle = \frac{1}{\sqrt{2}} \langle n | \left(a + a^\dagger\right) | n' \rangle = \sqrt{\frac{n}{2}}\delta_{n,n'+1} + \sqrt{\frac{n+1}{2}}\delta_{n,n'-1}$$

$$\langle n | p | n' \rangle = \frac{1}{\sqrt{2}i} \langle n | \left(a - a^\dagger\right) | n' \rangle = i\sqrt{\frac{n}{2}}\delta_{n,n'+1} = -i\sqrt{\frac{n+1}{2}}\delta_{n,n'-1}$$

Uncertainty

- In the basis of the energy eigenstates, the expectation values are

$$\langle p^2 \rangle = \langle q^2 \rangle = \langle H \rangle$$

$$\langle p \rangle = \langle q \rangle = 0$$

$$\Delta q = \sqrt{\langle q^2 \rangle} = \Delta p = \sqrt{n + \frac{1}{2}}$$

$$\Delta q \Delta p = n + \frac{1}{2}$$

wave functions

- Let ξ to be the eigenvalue of dimensionless coordinate q. The wave function can be expressed as

$$\langle \xi | n \rangle = \varphi_n(\xi)$$

- the operator p $\quad \langle \xi | p | n \rangle = \frac{1}{i} \frac{d}{d\xi} \varphi_n(\xi)$

$$\langle \xi | a | n \rangle = \frac{1}{\sqrt{2}} \langle \xi | q + ip | n \rangle = \frac{1}{\sqrt{2}} \left(\xi + \frac{d}{d\xi} \right) \varphi_n(\xi)$$

- the ground state $\quad a | 0 \rangle = 0$

$$\langle \xi | a | 0 \rangle = \frac{1}{\sqrt{2}} \left(\xi + \frac{d}{d\xi} \right) \varphi_0(\xi) = 0$$

Ground state

- In dimensionless variable

$$\left(\xi + \frac{d}{d\xi}\right)\varphi_0(\xi) = 0$$

$$\varphi_0(\xi) = \pi^{-\frac{1}{4}} \exp\left(-\frac{\xi^2}{2}\right)$$

- In physical variable

$$\varphi_0(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

- the dimension of wavefunction $\varphi_0(x) = \rightarrow L^{-\frac{1}{2}}$

excited states

- To construct the excited states by

$$\frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle = |n\rangle$$

$$\langle\xi|n\rangle = \frac{1}{\sqrt{n!}}\langle\xi|\left(a^\dagger\right)^n|0\rangle = \frac{1}{\sqrt{n!}}2^{-\frac{n}{2}}\left(\xi - \frac{d}{d\xi}\right)^n\varphi_0(\xi)$$

$$\varphi_n(\xi) = \frac{1}{\sqrt{n!}}\pi^{-\frac{1}{4}}2^{-\frac{n}{2}}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\frac{\xi^2}{2}}$$

$$\varphi_n(\xi) = \frac{1}{\sqrt{n!}}\pi^{-\frac{1}{4}}2^{-\frac{n}{2}}e^{\frac{\xi^2}{2}}\left(-\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

Here we used the identity

$$e^{\frac{\xi^2}{2}}\frac{d}{d\xi}\left(e^{-\frac{\xi^2}{2}}f\right) = \frac{d}{d\xi}f - \xi f = \left(\frac{d}{d\xi} - \xi\right)f$$

Hermite polynomials

$$H_n(\xi) = e^{\xi^2}\left(-\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

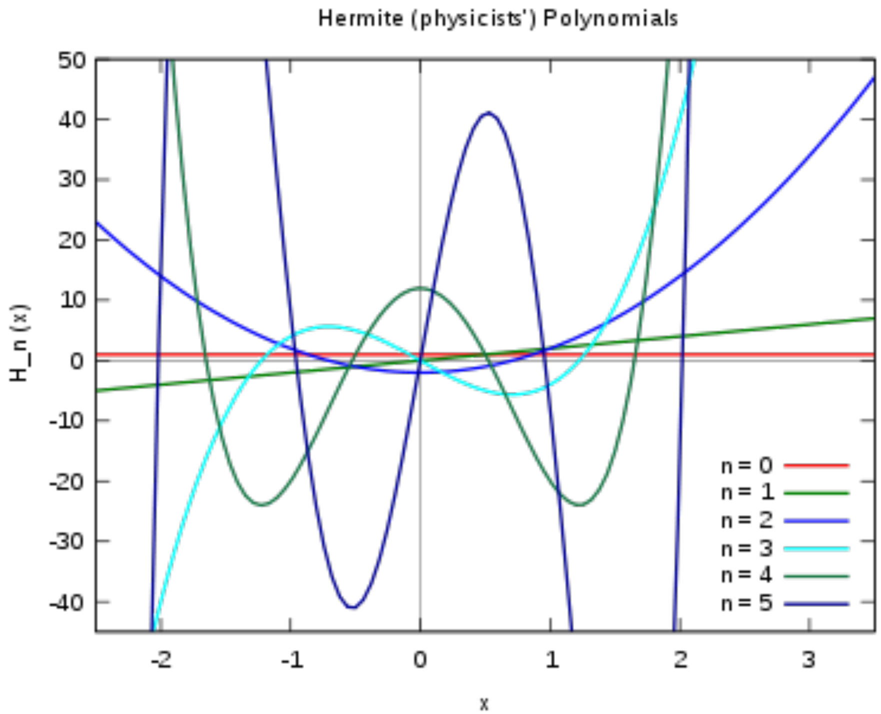
$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$



- Energy and parity can be specified simultaneously. Even or odd according to whether n is even or odd.

Displacement operator

- spatial displacement operator $e^{-ipx_0/\hbar}$

$$e^{ipx_0/\hbar} x e^{-ipx_0/\hbar} = x + x_0$$

- a displacement of *a* would be generated by the conjugate operator a^+. To construct a Unitary operator

$$D(z) = \exp\left(za^\dagger - z^* a\right)$$

- it produces a displacement of a through z:

$$D^\dagger(z) a D(z) = a + z$$

Baker–Campbell–Hausdorff formula

$$
\begin{aligned}
Z(X,Y) &= \log(\exp X \exp Y) \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] \\
&\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\
&\quad - \frac{1}{720}([[[[X,Y],Y],Y],Y] + [[[[Y,X],X],X],X]) \\
&\quad + \frac{1}{360}([[[[X,Y],Y],Y],X] + [[[[Y,X],X],X],Y]) \\
&\quad + \frac{1}{120}([[[[Y,X],Y],X],Y] + [[[[X,Y],X],Y],X]) + \cdots
\end{aligned}
$$

- If [A,B] commute with A and B, then

$$\ln\left(e^{A}e^{B}\right) = A + B + \frac{1}{2}\left[A,B\right]$$

$$D(z) = e^{za^\dagger - z^* a} = e^{za^\dagger} e^{-z^* a} e^{\frac{1}{2}\left[za^\dagger, z^* a\right]} = e^{-\frac{1}{2}|z|^2} e^{za^\dagger} e^{-z^* a}$$

$$D^\dagger(z) = e^{-\frac{1}{2}|z|^2} e^{-za^\dagger} e^{z^* a} = D^{-1}(z)$$

$$D^\dagger(z) a D(z) = e^{-|z|^2} e^{-za^\dagger} e^{z^* a} a e^{za^\dagger} e^{-z^* a}$$

$$= e^{-|z|^2} e^{-za^\dagger} e^{z^* a} e^{za^\dagger} a e^{-z^* a} + e^{-|z|^2} e^{-za^\dagger} e^{z^* a} \left[a, e^{za^\dagger}\right] e^{-z^* a}$$

$$= e^{-|z|^2} e^{-za^\dagger} e^{z^* a} e^{za^\dagger} e^{-z^* a} (a + z)$$

$$= D^\dagger(z) D(z)(a + z) = a + z$$

$$\left[a, e^{za^\dagger}\right] = z e^{za^\dagger}$$

Here we used the identity $\quad \left[a, F(a^\dagger)\right] = \dfrac{\partial F}{\partial a^\dagger}$

Displacement in phase space

- using the parametrization of

$$z = \frac{1}{\sqrt{2}} \left(q_0 + ip_0\right) \qquad |z|^2 = \frac{1}{2} \left(q_0^2 + p_0^2\right) = E_0$$

$$za^\dagger - z^* a = \frac{1}{2} \left(q_0 + ip_0\right) \left(q - ip\right) - h.c. = ip_0 q - iq_0 p$$

- a displacement in phase space

$$D = \exp\left(ip_0 q - iq_0 p\right) = e^{-\frac{i}{2} p_0 q_0} e^{ip_0 q} e^{-iq_0 p}$$

$$D^\dagger q D = q + q_0 \qquad D^\dagger p D = p + p_0$$

coherent state

- translate the ground state by z

$$|z\rangle = D(z)|0\rangle$$

- it is an eigenstate of the non-Hermitian operator *a*

$$a|z\rangle = aD(z)|0\rangle = D(z)(a+z)|0\rangle = z|z\rangle$$

- The expansion of the coherent state in terms of energy eigenstates

$$a|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

Poisson distribution

- the probabilities are given by the Poisson distribution

$$p_n(z) = \frac{e^{-|z|^2}|z|^{2n}}{n!} = \frac{e^{-E_0}E_0^n}{n!}$$

- The mean value of *N* is

$$\langle N \rangle = \sum_n n p_n = e^{-E_0} E_0 \frac{d}{dE_0} \sum_n \frac{E_0^n}{n!} = E_0$$

$$\sum_n \frac{E_0^n}{n!} = e^{E_0}$$

- The mean value of *N*(*N*-1)

$$\langle N(N-1) \rangle = \sum_n n(n-1) p_n = e^{-E_0} E_0^2 \frac{d^2}{dE_0^2} \sum_n \frac{E_0^n}{n!} = E_0^2$$

Energy fluctuation

- fluctuation in N

$$\Delta N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \langle N(N-1) \rangle + \langle N \rangle - \langle N^2 \rangle = E_0$$

$$\Delta N = \sqrt{E_0}$$

- when the amplitude of the oscillating wave packet is large compared to the size of the ground state, the energy eigenstates that contribute significantly are sharply peaked around the state with $n = E_0$.