superconductivity

Classical electrodynamics

Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Scalar and vector potentials

Maxwell equations

$$\nabla \cdot \mathbf{B} = 0$$

The magnetic field can be the curl of a vector field $\mathbf{B} = \nabla \times \mathbf{A}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

The curl of a gradient field is zero

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

Hamilton formalism

In absence of electromagnetic fields, the particle Hamiltonian is

$$H = \frac{p^2}{2m} + V\left(x\right)$$

The equations of motion are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

We may get

$$\frac{dx_i}{dt} = \frac{p_i}{m} = v_i$$

$$\frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i}$$

$$m\frac{d^2 x_i}{dt} = -\frac{\partial V}{\partial x_i}$$

Lorentz force

The charged particle experiences a force from the EM fields

$$m\frac{d^2}{dt^2}\mathbf{x} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

We could take account in its contribution by considering the hamiltonian

$$H = \frac{\left(\mathbf{p} - q\mathbf{A}\right)^2}{2m} + q\phi$$

p is called canonical momentum

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

Derivation

Consider the equations of motion

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{\left(p_i - qA_i\right)}{m}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = \frac{q}{m}\sum_k \left(p_k - qA_k\right)\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

Combine together

$$\begin{aligned}
m\frac{d^2 x_i}{dt^2} &= \frac{dp_i}{dt} - q\frac{dA_i}{dt} \\
&= \frac{dp_i}{dt} - q\frac{\partial A_i}{\partial t} - q\sum_k \frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt} \\
&= \frac{q}{m}\sum_k \left(p_k - qA_k\right)\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} - q\frac{\partial A_i}{\partial t} - q\sum_k \frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt} \\
&= q\sum_k \frac{dx_k}{dt}\frac{\partial A_k}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} - q\frac{\partial A_i}{\partial t} - q\sum_k \frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt} \\
&= q\sum_k \frac{dx_k}{dt}\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right) + q\left(-\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t}\right) \\
&= q\left(\mathbf{v}\times\mathbf{B}\right)_i + qE_i
\end{aligned}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Quantum mechanics for charged particles

hamiltonian

$$H = \frac{\left(\mathbf{p} - q\mathbf{A}\right)^2}{2m} + q\phi$$

operator

$$\mathbf{p} = \frac{\hbar}{i}\nabla$$

Equation of motion

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi$$

Gauge transformation

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

There are many solutions for the scalar and vector fields

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi$$

$$\phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t}$$

This is called gauge transformation

Transform on wavefunctions

hamiltonian $$H' = \frac{\left(\mathbf{p} - q\mathbf{A} - q\nabla\chi\right)^2}{2m} + q\phi - q\frac{\partial\chi}{\partial t}$$

The wavefunction will become $$\psi \to \psi' = e^{i\lambda}\psi$$

λ is a function of position

$$i\hbar\frac{\partial}{\partial t}\psi' = H'\psi'$$

derivation

$$i\hbar\frac{\partial}{\partial t}\psi' = i\hbar\frac{\partial}{\partial t}\left(e^{i\lambda}\psi\right) = i\hbar e^{i\lambda}\frac{\partial}{\partial t}\psi - \hbar\frac{\partial\lambda}{\partial t}e^{i\lambda}\psi$$

$$\left(\mathbf{p} - q\mathbf{A} - q\nabla\chi\right)\psi' = \left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi\right)e^{i\lambda}\psi = e^{i\lambda}\left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi + \hbar\nabla\lambda\right)\psi$$

$$\left(\mathbf{p} - q\mathbf{A} - q\nabla\chi\right)^2\psi' = \left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi\right)e^{i\lambda}\left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi + \hbar\nabla\lambda\right)\psi$$

$$= e^{i\lambda}\left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi + \hbar\nabla\lambda\right)^2\psi$$

$$H'\psi' = e^{i\lambda}\left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\nabla\chi + \hbar\nabla\lambda\right)^2\psi + q\phi e^{i\lambda}\psi - q\frac{\partial\chi}{\partial t}e^{i\lambda}\psi$$

If set $\hbar\nabla\lambda = q\nabla\chi$

$$i\hbar\frac{\partial}{\partial t}\psi' = e^{i\lambda}H\psi - \hbar\frac{\partial\lambda}{\partial t}e^{i\lambda}\psi = H'\psi'$$

$$\psi' = e^{\frac{i}{\hbar}q\chi}\psi$$

Probability density and current density

The probability density can be written as

$$\rho = |\psi|^2$$

From continuity equation, the current density(or probability flux) is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}$$

The derivative of P can be calculated as

$$\frac{\partial}{\partial t}|\psi|^2 = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi = \frac{1}{i\hbar}\left[\psi^* H\psi - \psi\left(H\psi\right)^*\right]$$

derivation

$$H\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi + q\phi\psi \qquad \left(H\psi\right)^* = \frac{1}{2m}\left(-\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi^* + q\phi\psi^*$$

If A=0

$$\psi^* H\psi = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi$$

$$\psi\left(H\psi\right)^* = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi^*\psi$$

$$\frac{\partial}{\partial t}\left|\psi\right|^2 = -\frac{\hbar}{2mi}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = -\frac{\hbar}{2mi}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

derivation

$$H\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi + q\phi\psi \qquad \left(H\psi\right)^* = \frac{1}{2m}\left(-\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi^* + q\phi\psi^*$$

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{i}\nabla\psi - q\mathbf{A}\psi\right)$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2\psi - 2\frac{\hbar}{i}q\mathbf{A}\cdot\nabla\psi - q\frac{\hbar}{i}\left(\nabla\cdot\mathbf{A}\right)\psi + q^2\mathbf{A}^2\psi\right]$$

$$\frac{1}{2m}\left(-\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 \psi^* = \frac{1}{2m}\left[-\hbar^2\nabla^2\psi^* + 2\frac{\hbar}{i}q\mathbf{A}\cdot\nabla\psi^* + q\frac{\hbar}{i}\left(\nabla\cdot\mathbf{A}\right)\psi^* + q^2\mathbf{A}^2\psi^*\right]$$

$$\psi^* H\psi - \psi\left(H\psi\right)^* = \frac{1}{2m}\left[-\hbar^2\psi^*\nabla^2\psi - 2\frac{\hbar}{i}q\mathbf{A}\cdot\psi^*\nabla\psi - q\frac{\hbar}{i}\left(\nabla\cdot\mathbf{A}\right)\psi^*\psi + q^2\mathbf{A}^2\psi^*\psi\right]$$

$$-\frac{1}{2m}\left[-\hbar^2\psi\nabla^2\psi^* + 2\frac{\hbar}{i}q\mathbf{A}\cdot\psi\nabla\psi^* + q\frac{\hbar}{i}\left(\nabla\cdot\mathbf{A}\right)\psi^*\psi + q^2\mathbf{A}^2\psi^*\psi\right]$$

$$= \frac{1}{2m}\left[-\hbar^2\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) - 2q\frac{\hbar}{i}\mathbf{A}\cdot\left(\psi^*\nabla\psi + \psi\nabla\psi^*\right) - 2q\frac{\hbar}{i}\left(\nabla\cdot\mathbf{A}\right)\psi^*\psi\right]$$

derivation

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* = \nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$\mathbf{A}\cdot\left(\psi^*\nabla\psi + \psi\nabla\psi^*\right) + \left(\nabla\cdot\mathbf{A}\right)\psi^*\psi = \nabla\cdot\left(\mathbf{A}\psi^*\psi\right)$$

$$\begin{aligned}\psi^* H\psi - \psi\left(H\psi\right)^* &= \frac{1}{2m}\left[-\hbar^2\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - 2q\frac{\hbar}{i}\nabla\cdot\left(\mathbf{A}\psi^*\psi\right)\right] \\ &= \frac{\hbar}{2mi}\nabla\cdot\left[\frac{\hbar}{i}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - 2q\mathbf{A}\psi^*\psi\right] \\ &= \frac{\hbar}{2mi}\nabla\cdot\left[\psi^*\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi - \psi\left(\frac{\hbar}{i}\nabla + q\mathbf{A}\right)\psi^*\right]\end{aligned}$$

$$\mathbf{j} = \frac{1}{2m}\left[\psi^*\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi - \psi\left(\frac{\hbar}{i}\nabla + q\mathbf{A}\right)\psi^*\right]$$

Wave function phase

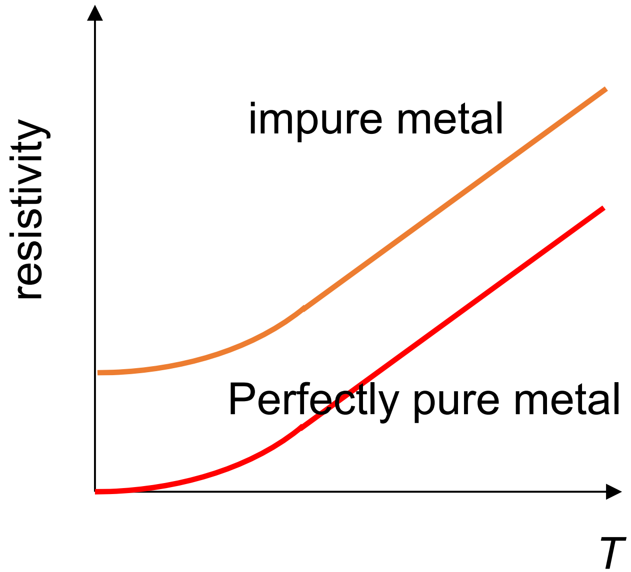
We may parametrize the wavefunction as

$$\psi = \sqrt{\rho}e^{i\theta} \qquad\qquad \rho = |\psi|^2$$

$$\begin{aligned}
\mathbf{j} &= \frac{1}{2m}\left[\sqrt{\rho}e^{-i\theta}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\sqrt{\rho}e^{i\theta} - \sqrt{\rho}e^{i\theta}\left(\frac{\hbar}{i}\nabla + q\mathbf{A}\right)\sqrt{\rho}e^{-i\theta}\right] \\
&= \frac{1}{2m}\left[\sqrt{\rho}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\sqrt{\rho} + \hbar\nabla\theta\rho - \sqrt{\rho}\left(\frac{\hbar}{i}\nabla + q\mathbf{A}\right)\sqrt{\rho} + \hbar\nabla\theta\rho\right] \\
&= \frac{1}{m}\left(\hbar\nabla\theta - q\mathbf{A}\right)\rho = \rho\mathbf{v}
\end{aligned}$$

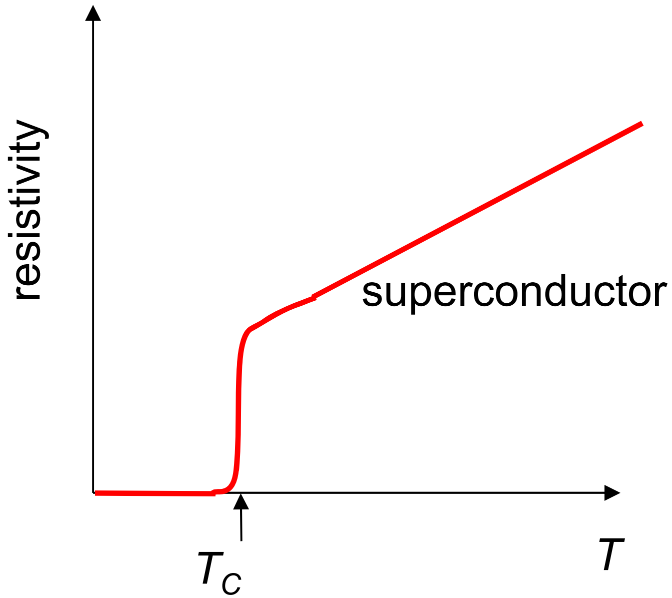
Velocity operator: $\mathbf{v} = \frac{1}{m}\left(\hbar\nabla\theta - q\mathbf{A}\right)$

Zero resistance



Residual resistivity

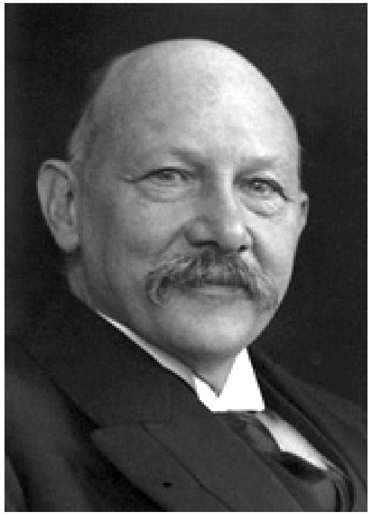
ρ_0



Critical temperature

transition temperature

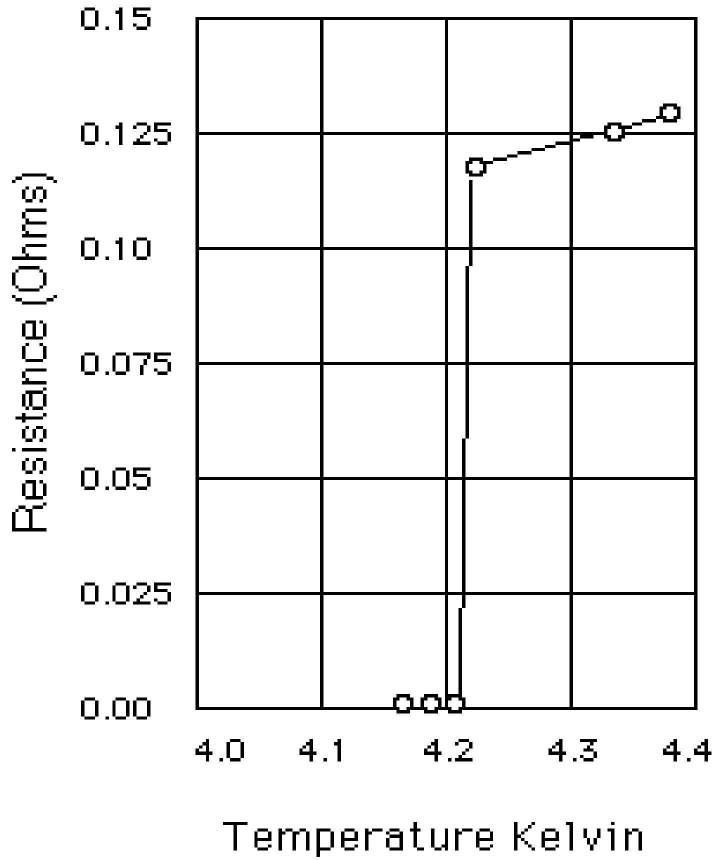
Discovery of superconductivity



H. Kamerlingh Onnes
(Leiden University)

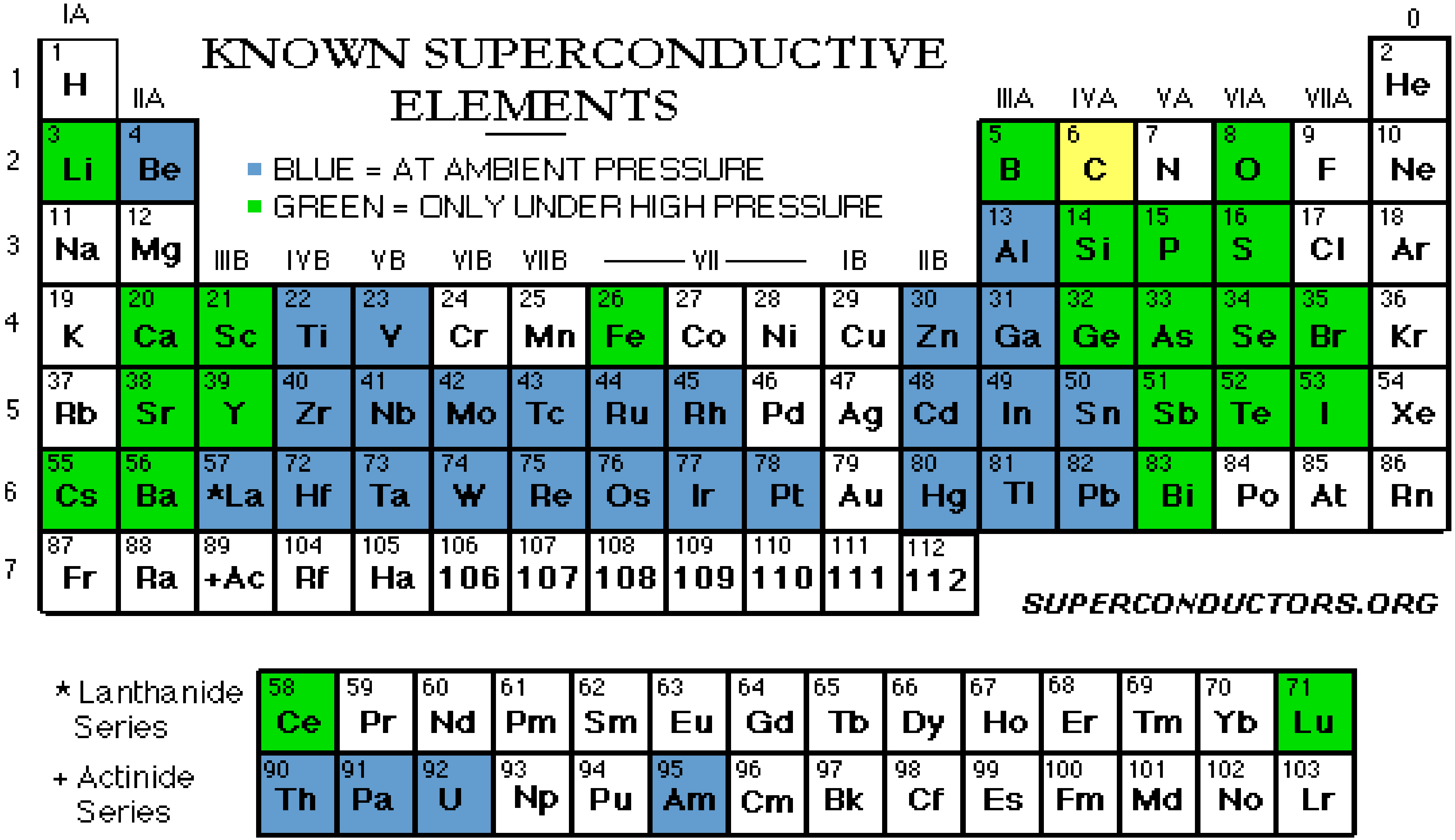
-first to liquify helium (1908),
-Nobel prize in 1913,
-discovered superconductivity in 1911

"Mercury has passed into a new state, which on account of its extraordinary electrical properties may be called the superconductive state".



H. K. Onnes, Commun. Phys. Lab.12,120, (1911)

Known superconductive elements



Transition temperatures

Nb	9.25K	BCC(Type 2)
Tc	7.80 K	HEX(Type 2)
Lead (Pb)	7.196 K	FCC
V	5.40 K	BCC(Type 2)
Lanthanum (La)	4.88 K	HEX
Tantalum(Ta)	4.47 K	BCC
Mercury (Hg)	4.15 K	RHL
Tin (Sn)	3.72 K	TET
Indium (In)	3.41 K	TET
Thallium (Tl)	2.38 K	HEX
Rhenium (Re)	1.697 K	HEX
Protactinium (Pa)	1.40 K	TET
Thorium (Th)	1.38 K	FCC
Aluminum (Al)	1.175 K	FCC
Gallium (Ga)	1.083 K	ORC
Molybdenum (Mo)	0.915 K	BCC
Zinc (Zn)	0.85 K	HEX
Osmium (Os)	0.66 K	HEX
Zirconium (Zr)	0.61 K	HEX
Americium (Am)	0.60 K	HEX
Cadmium (Cd)	0.517 K	HEX
Ruthenium (Ru)	0.49 K	HEX
Titanium (Ti)	0.40 K	HEX
Uranium (U)	0.20 K	ORC
Hafnium (Hf)	0.128 K	HEX
Iridium (Ir)	0.1125 K	FCC
Beryllium (Be)	0.023 K	HEX
Tungsten (W)	0.0154 K	BCC
Platinum (Pt)*	0.0019 K	FCC
Rhodium (Rh)	0.000325 K	FCC

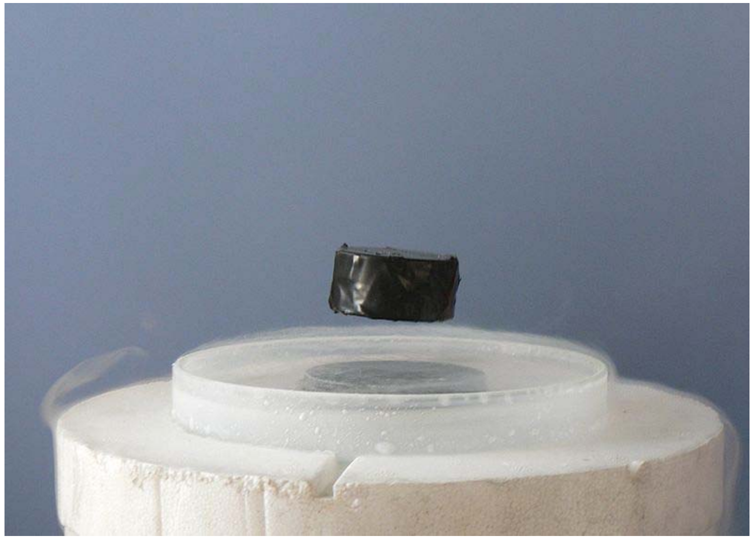
*compacted powder

MgB_2	39K
Nb_3Ge	23.2K

Meissner effect

Walter Meissner

Robert Ochsenfeld



@Physikalisch-Technische Reichsanstalt, Germany
1933

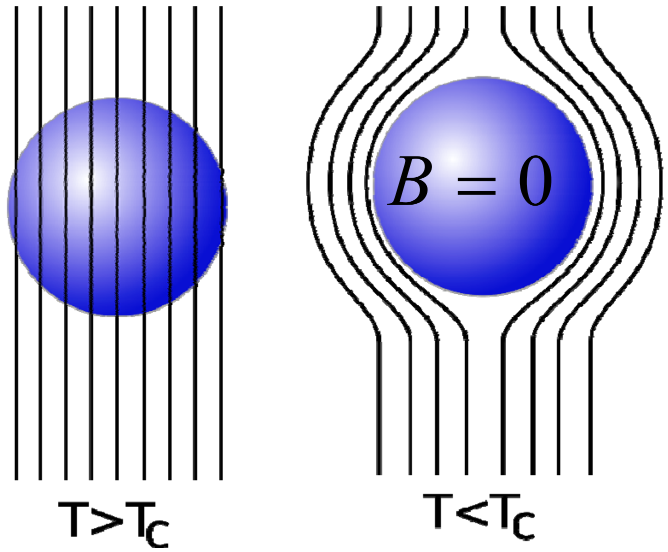
Perfect diamagnetism

Superconductor

Perfect diamagnetism:

Flux exclusion: zero field cooled

Flux expulsion: field cooled



Perfect conductor

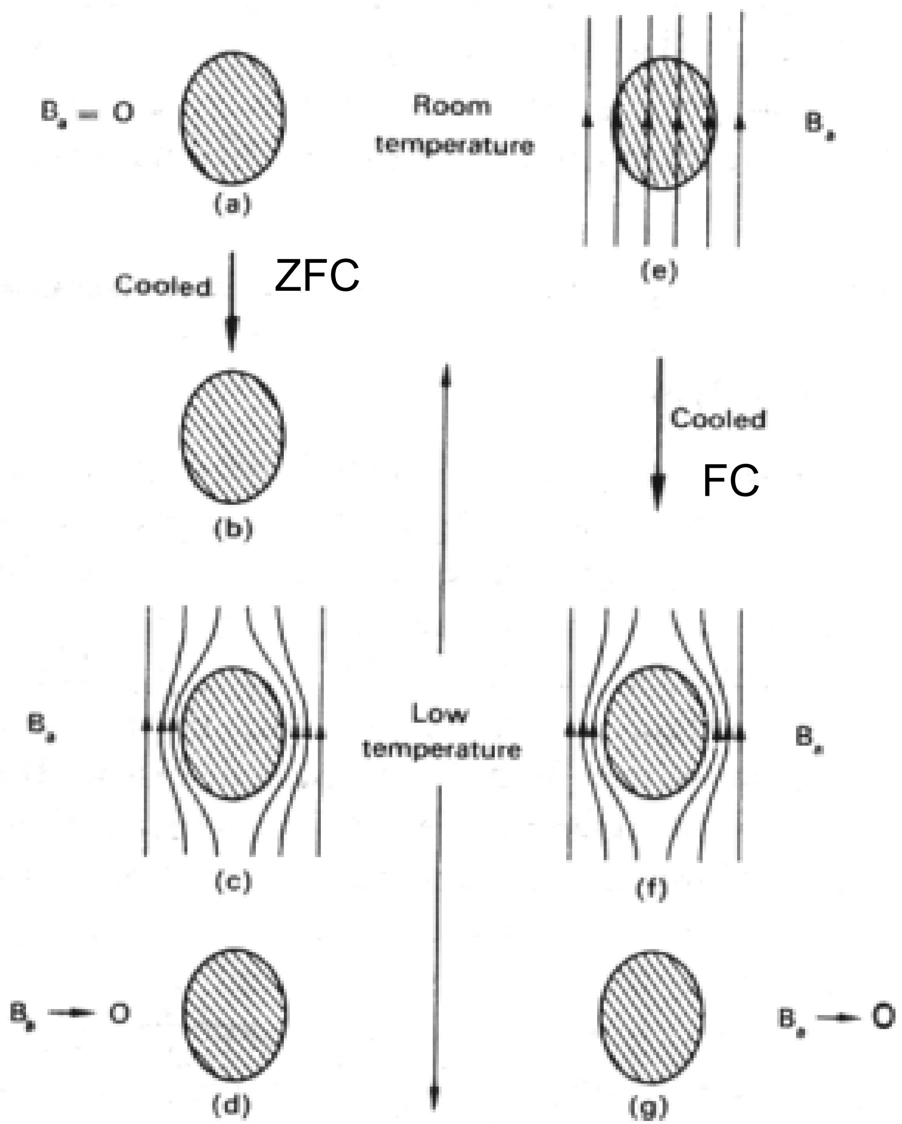
Constant magnetic flux

$$\dot{B} = 0$$

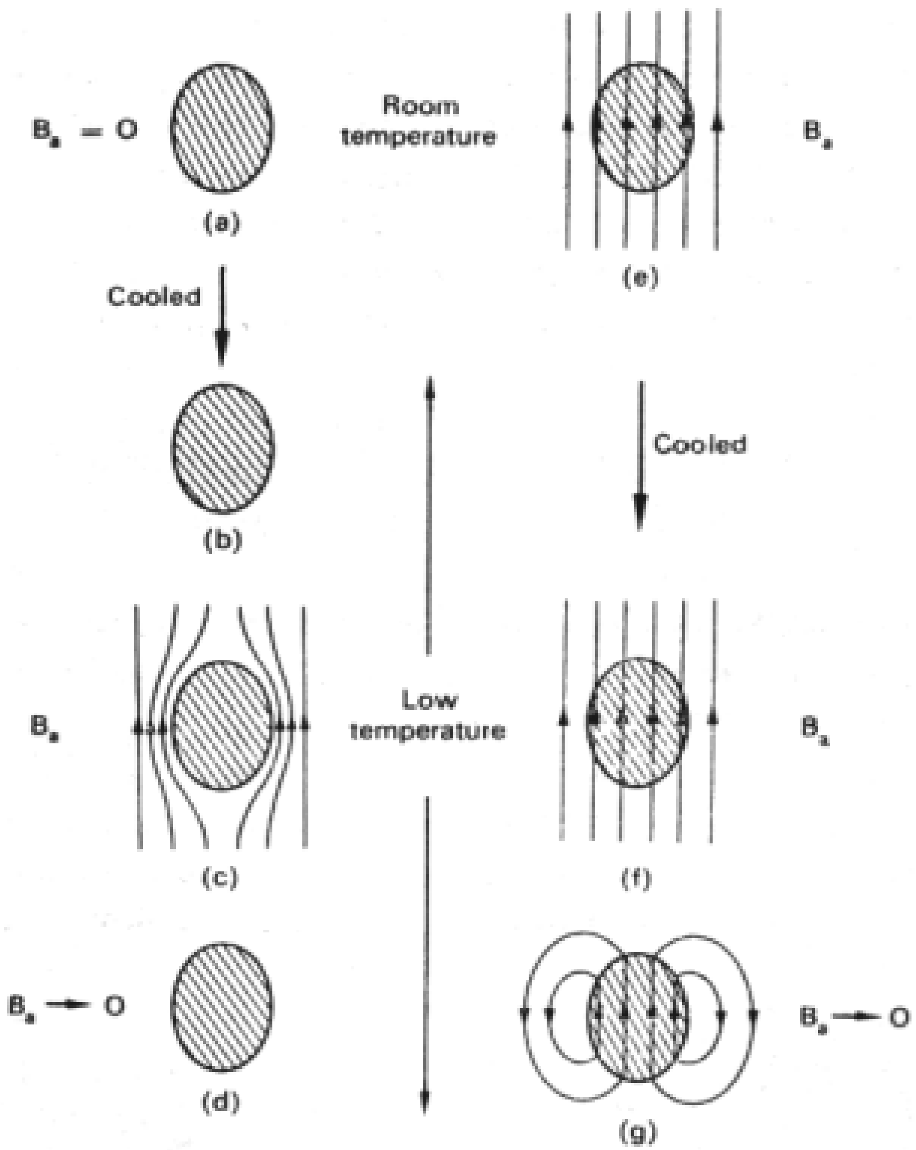
Recall the result deduced in page 14

Walther Meissner and Robert Ochsenfeld 1933

superconductor



Perfect conductor



ac conductivity of a conductor

In Drude model, the mean-free time approximation assumes

$$\frac{dp}{dt} = F_{ext} - \frac{p}{\tau} = eE - \frac{p}{\tau}$$

Current density $J = nev = \dfrac{nep}{m}$ $\qquad \dfrac{dJ}{dt} = \dfrac{ne^2 E}{m} - \dfrac{J}{\tau}$

At frequency ω $\qquad \dfrac{dJ}{dt} = -i\omega J$ $\qquad \left(1 - i\omega\tau\right) J = \dfrac{ne^2 \tau E}{m}$

Ohm's law $\qquad J\left(\omega\right) = \sigma\left(\omega\right) E\left(\omega\right)$

AC conductivity $\qquad \sigma(\omega) = \dfrac{\sigma_0}{1 - i\omega\tau}$ $\qquad \sigma_0 = \dfrac{ne^2\tau}{m}$

Result of a perfect conductor

For superelectons, $\tau \rightarrow \infty$

AC conductivity $$\sigma(\omega) = i\frac{ne^2}{m\omega}$$ $$\operatorname{Im}\sigma(\omega) \propto \frac{1}{\omega}$$

Inductive impedance

$$\frac{\partial J}{\partial t} = \frac{ne^2}{m}E$$

Maxwell equation

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla \times J + \frac{ne^2}{m}B\right) = 0$$

Maxwell equation

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\nabla \times J + \frac{ne^2}{m}B\right) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla \times \nabla \times B + \mu_0 \frac{ne^2}{m}B\right) = 0$$

use $\nabla \times \nabla \times B = \nabla\left(\nabla \bullet B\right) - \nabla^2 B = -\nabla^2 B$

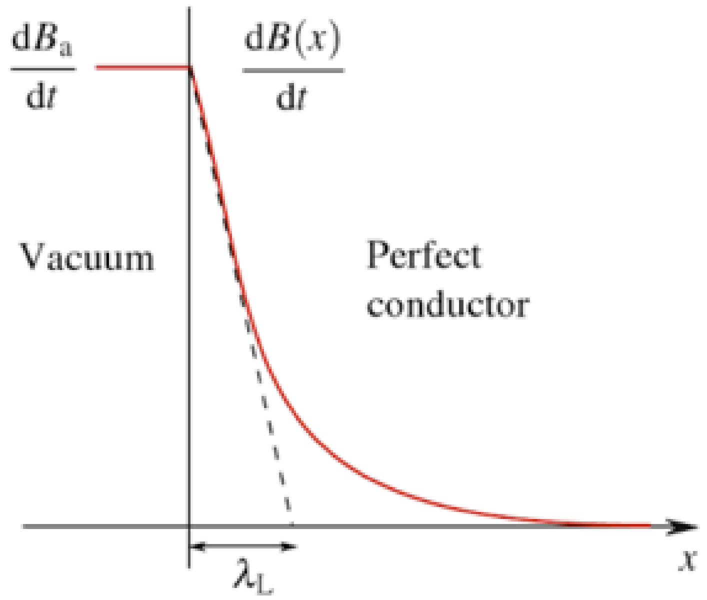
$$\Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla^2 B - \mu_0 \frac{ne^2}{m}B\right) = 0$$

$$\nabla^2 \dot{B} - \frac{1}{\lambda^2}\dot{B} = 0$$

Constant magnetic flux

$$\dot{B} = \dot{B}_a \exp\left(\frac{-x}{\lambda}\right)$$

$$\lambda = \sqrt{\frac{m}{\mu_0 ne^2}}$$



The London theory

Fritz and Heinz London 1935
To match the experimental result

$$\nabla^2 \dot{B} - \frac{1}{\lambda^2}\dot{B} = 0 \quad \Longrightarrow \quad \boxed{\nabla^2 B - \frac{1}{\lambda^2} B = 0}$$

$$\boxed{\nabla \times J + \frac{ne^2}{m} B = 0} \qquad \boxed{\frac{\partial J}{\partial t} = \frac{ne^2}{m} E}$$

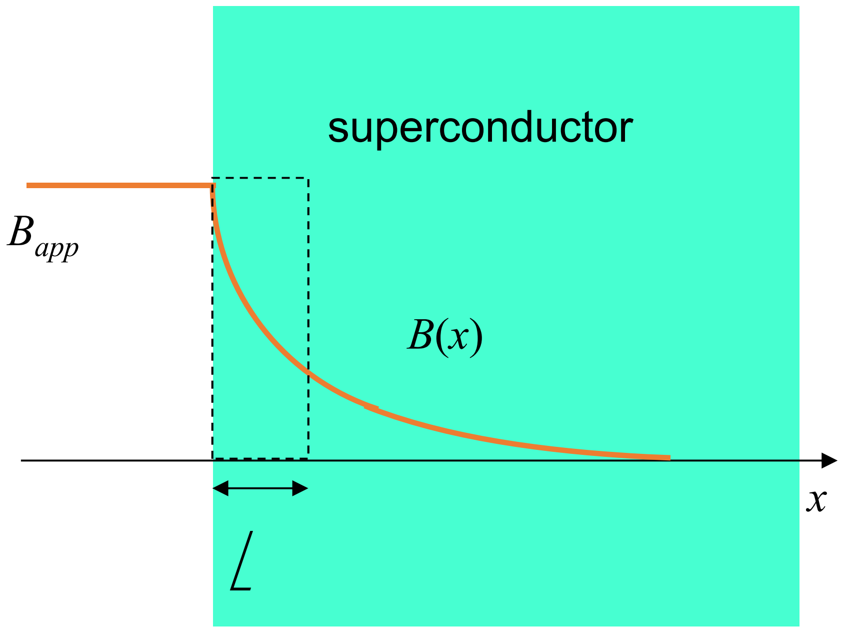
These equations are called London equations

London penetration depth

To solve *B*(*x*) and apply the boundary condition

$\Longrightarrow$

$$B = B_a \exp\left(\frac{-x}{\lambda_L}\right)$$



λ_L is called London penetration depth

Value of London penetration depth

$$\lambda = \sqrt{\frac{m}{\mu_0 n e^2}}$$

Electron charge~10^{-19} C

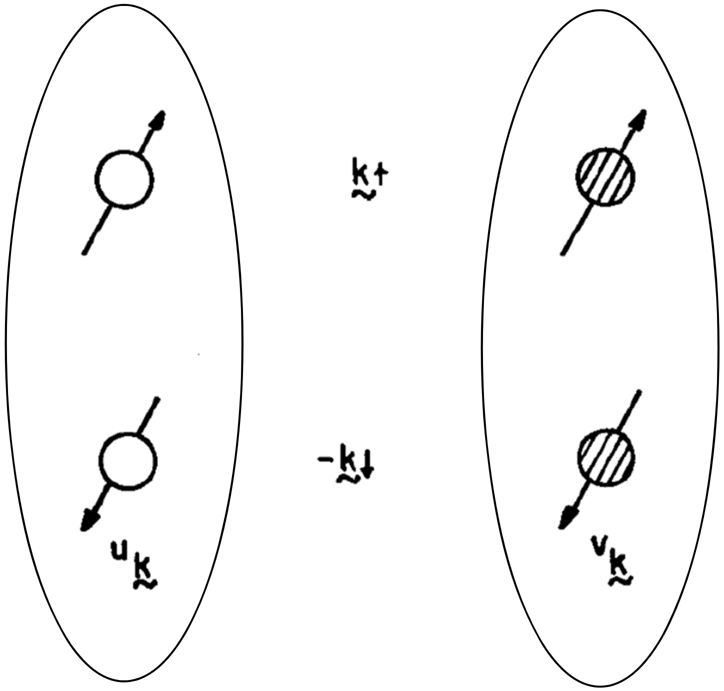
Electron mass ~ 10^{-30} kg

$\mu_0 = 4\pi \times 10^{-7}$ H/m

Electron number density ~10^{28} m^{-3}

$\lambda \sim 10^{-7}$ m

Coherence length



Occupied unoccupied

Electron pairing:
Cooper pairs

The coherent length can be viewed as the size of the Cooper pair

http://nobelprize.org/physics/laureates/1972/cooper-lecture.html

Ginzburg-Landau theory

1. A macroscopic theory
2. A phenomenological theory
3. A quantum theory ⟺ London theory is classical

Introduction of pseudo wave function $\Psi(\mathbf{r})$

$\left|\Psi(\mathbf{r})\right|^2$ is the local density of superconducting electrons

$$\left|\Psi(\mathbf{r})\right|^2 = n_s(\mathbf{r})$$

The free energy density

The difference of free energy density for normal state and superconducting state can be written as powers of

$$|\Psi|^2 \quad \text{and} \quad |\nabla\Psi|^2$$

potential energy

Kinetic energy

Ginzburg-Landau free energy density at zero field

$$g_s = g_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\frac{\hbar}{i}\nabla\Psi\right|^2$$

2nd order phase transition

Quantum mechanics

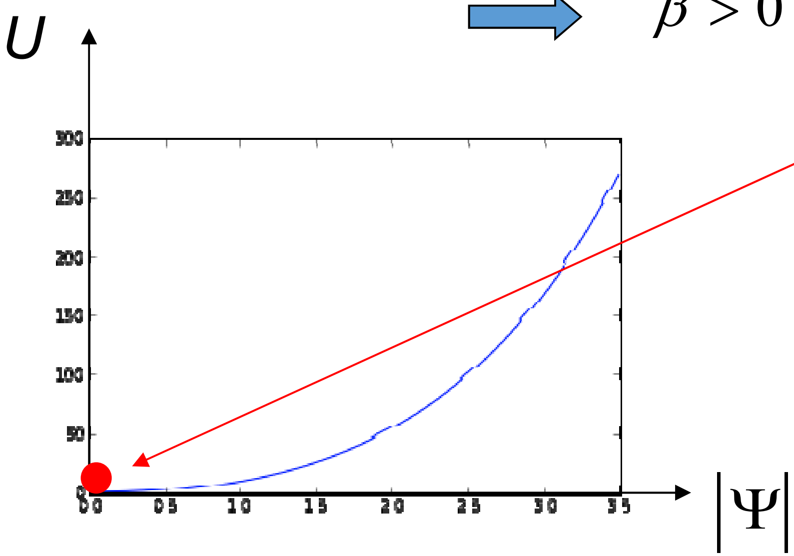
2nd order phase transition

Potential energy $$U = \alpha\left|\Psi\right|^2 + \frac{\beta}{2}\left|\Psi\right|^4$$

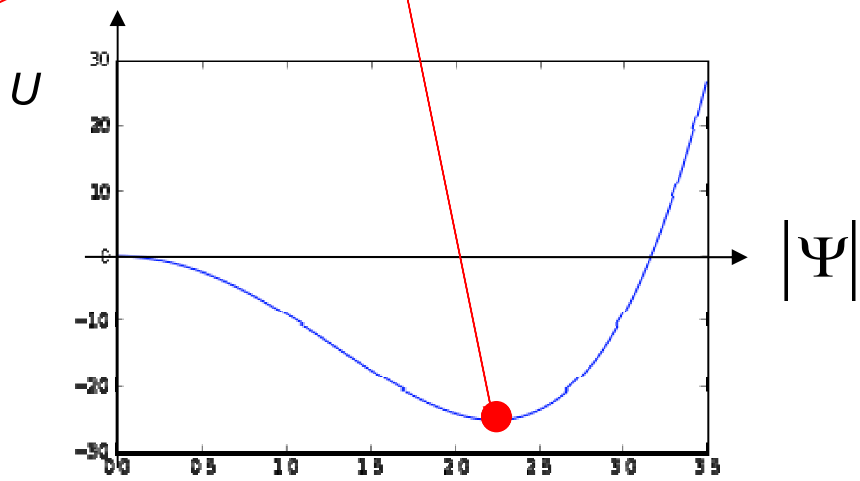
A reasonable theory is bounded, i. e. $$U\left(\left|\Psi\right| \to \infty\right) \to \infty$$

$$\Rightarrow \quad \beta > 0$$

Classical solutions

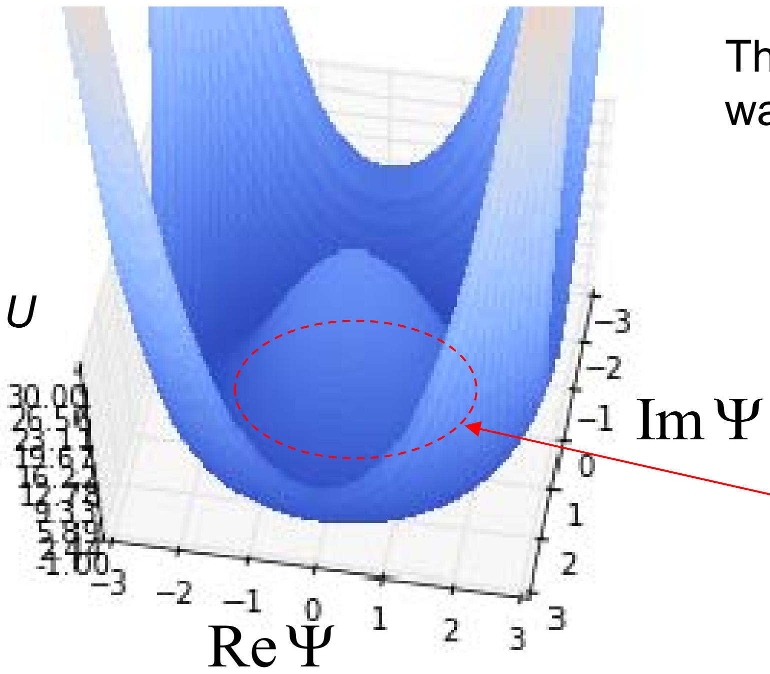


$\alpha > 0 \qquad \left|\Psi\right| = 0$



$\alpha < 0 \qquad \left|\Psi\right| \neq 0$

Spontaneous symmetry breaking



The phase symmetry of the ground state wave function is broken

$$\Psi = |\Psi| e^{i\varphi}$$

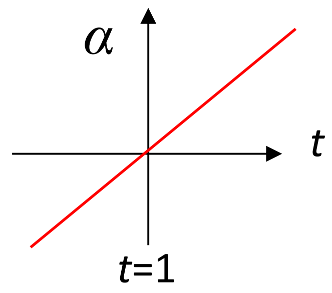
$$|\Psi|^2 = \Psi_\infty^2 = -\frac{\alpha}{\beta}$$

$\alpha > 0 \quad |\Psi| = 0$	$\alpha = 0$	$\alpha < 0 \quad |\Psi| \neq 0$
Normal state	**Critical point**	**superconducting state**

$|\Psi|^2$ density of superconducting electrons

The meaning of α

The superconducting critical point is $\alpha = 0$



→

$\alpha > 0$	$\alpha = 0$	$\alpha < 0$
$T > T_c$	$T = T_c$	$T < T_c$

Near the critical point, $\alpha = \alpha'(t-1)$

$$t = \frac{T}{T_c}$$

If β is regular near T_c then $|\Psi|^2 = -\frac{\alpha'}{\beta_c}(t-1)$

The London penetration depth is $\lambda_L^2 = \frac{m}{\mu_0 n_s e^2}$

$$\Rightarrow \quad \lambda_L \propto \left(\frac{1}{n_s}\right)^{\frac{1}{2}} \propto \frac{1}{(1-t)^{1/2}}$$

Consistent with the observation

$$\frac{\lambda_L(T)}{\lambda_L(0)} = \frac{1}{\left(1-t^4\right)^{1/2}}$$

Magnetic field contribution

at non zero field, there are two modifications

$$\mathbf{p} \rightarrow \mathbf{p} - e^{*}\mathbf{A}$$

The vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\Delta g = \frac{1}{2}\mu_0 H^2$$

For perfect diamagnetism

$$\Delta g\left(H_a\right) = -\mu_0 \int_0^{H_a} M dH_a$$

The canonical momentum

The first modification is to include the hamiltonian of a charged particle in a magnetic field

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

For a charged paticle, $$m\mathbf{v}(t) = m\mathbf{v}(0) + q\int_0^t \mathbf{E}\,dt$$
$$= m\mathbf{v}(0) - q\mathbf{A}$$

$m\mathbf{v}(t) + q\mathbf{A} = m\mathbf{v}(0)$ is conserved in the magnetic field

The canonical momentum is chosen as $\mathbf{p}_{\mathrm{canonical}} = m\mathbf{v} + q\mathbf{A}$

The kinetic energy is $$\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2m}\left(\mathbf{p}_{\mathrm{canonical}} - q\mathbf{A}\right)^2$$

Gauge transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \chi$$

$$\phi \to \phi' = \phi - \frac{\partial}{\partial t}\chi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The physics is unchanged

The phase of the particle wave function will be changed by a phase factor

$$\Psi(\mathbf{r}) \to \Psi'(\mathbf{r}) = \Psi(\mathbf{r})\exp\left(\frac{ie}{\hbar}\chi\right)$$

$$\left\langle \Psi H \Psi \right\rangle = \left\langle \Psi' H' \Psi' \right\rangle$$

GL theory is gauge-invariant.

The meaning of $|\Psi|^2$

Energy density

$$\frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - e^* A\right)\Psi\right|^2 = \frac{1}{2m^*}\left|\left(\underbrace{\frac{\hbar}{i}\nabla|\Psi|}_{\text{Imag part}} + \underbrace{\hbar|\Psi|\nabla\varphi - e^* A|\Psi|}_{\text{Real part}}\right)e^{i\varphi}\right|^2$$

with $\Psi = |\Psi| e^{i\varphi}$

$$= \frac{1}{2m^*}\left\{\hbar^2\left(\nabla|\Psi|\right)^2 + \left(\hbar\nabla\varphi - e^* A\right)^2|\Psi|^2\right\}$$

•The first term arises when the number density n_s has a non- zero gradient, for example near the N-S boundary

(the length scale is coherent length ξ, and in type I SC, ξ<<λ)

•The second term is the kinetic term associated with the supercurrent. If the phase is constant of position, it gives

$$= \frac{e^{*2} A^2 |\Psi|^2}{2m^*}$$

GL differential eqns

The solution for minimizing g_s in absence of the field, boundary and current is

$$\Psi = \Psi_\infty$$

In general cases, the wavefunction can be written as

$$\Psi = \Psi(\mathbf{r})$$

By variational method

$$\delta \int_V g_S dV = 0$$

We have

$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)^2\Psi = 0 \quad \text{(1st eq)}$$

$$\mathbf{J} = \frac{e^*\hbar}{2m^*i}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{e^{*2}}{m^*}\mathbf{A}|\Psi|^2 \quad \text{(2nd eq)}$$

$$= \frac{e^*}{m^*}\left(\hbar\nabla\varphi - e^*\mathbf{A}\right)|\Psi|^2 = e^*|\Psi|^2\mathbf{v}_S$$

GL coherent length

At zero field, H=0

$$\mathbf{J} = 0 \qquad \Psi^* \nabla \Psi - \Psi \nabla \Psi^* = 0 \qquad \text{and}$$

$$\nabla \varphi = 0$$ Superconducting phase is constant of position

(GL eq 1)

$$\alpha \Psi + \beta |\Psi|^2 \Psi - \frac{\hbar^2}{2m^*} \nabla^2 \Psi = 0$$

$$f \equiv \frac{\Psi}{\Psi_\infty}$$

$$\Psi_\infty \equiv -\frac{\alpha}{\beta}$$

In 1D system

$$-\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} f + \alpha f + \beta |\Psi_\infty|^2 f^3 = 0$$

$$-\frac{\hbar^2}{2m^* |\alpha|} \frac{d^2}{dx^2} f + f - f^3 = 0$$

Dimension=[L^2]

A length scale can be defined

$$\frac{\hbar^2}{2m^*|\alpha|} = \xi^2$$

$$-\xi^2 \frac{d^2}{dx^2} f + f - f^3 = 0$$

since

$$\alpha \propto 1-t \qquad \xi^2 \propto \frac{1}{1-t}$$

Consider the situation that *f*~1 (deep in the SC)

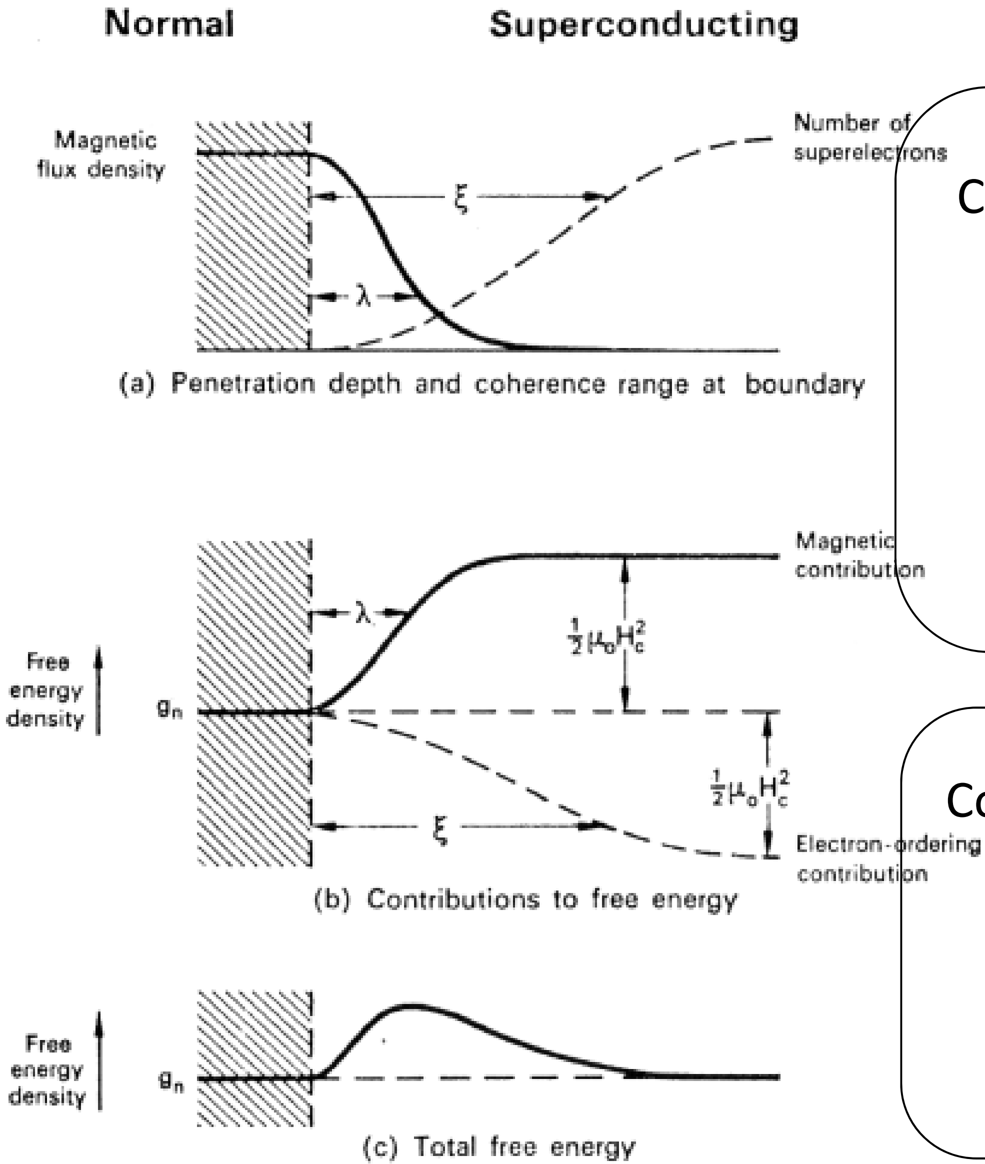
We can expand the GL eq:

$$f = 1-g \qquad g \simeq 0$$

$$\xi^2 \frac{d^2}{dx^2} g + (1-g) - (1-g)^3 = 0$$

$$\xi^2 g'' + 2g = 0$$

$$g(x) \simeq e^{\pm\sqrt{2}x/\xi}$$



Coherence length > penetration depth

surface energy > 0

Type I SC

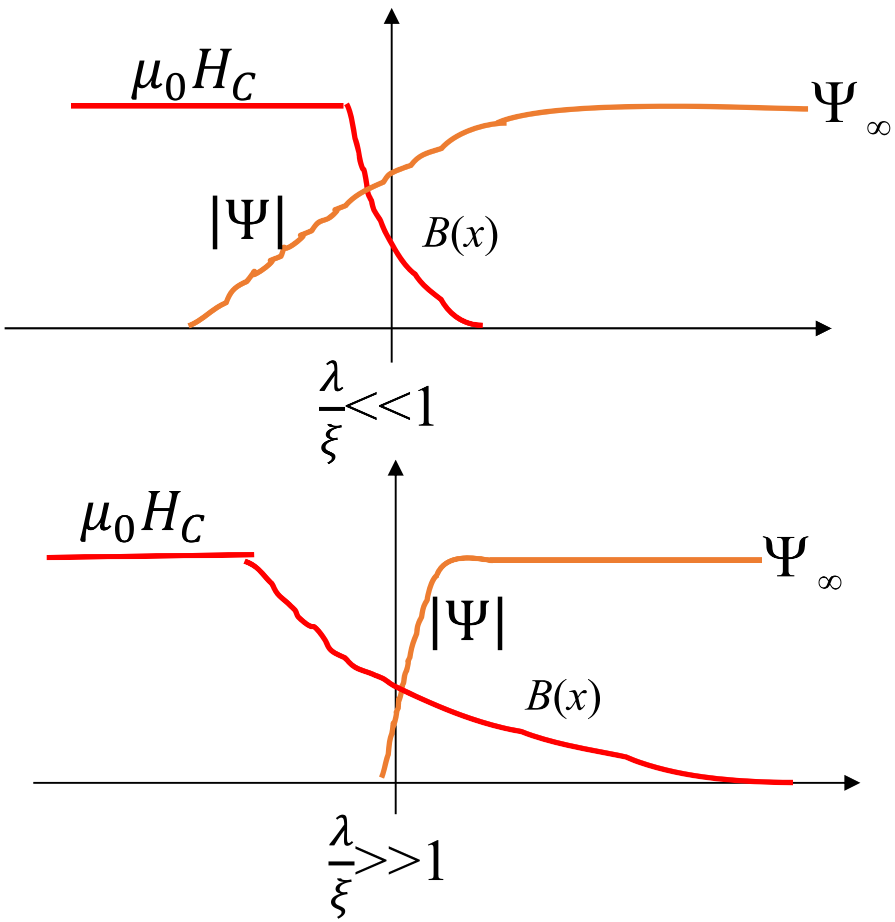
Coherence length < penetration depth

surface energy < 0

Type II SC

Domain wall energy

Competition of the two length scales



Fluxoid

Introduced by F. London

$$\Phi' = \Phi + \mu_0 \lambda^2 \oint \mathbf{J}_s \cdot d\mathbf{l}$$

$$= \Phi + \frac{m^*}{e^*} \oint \mathbf{v}_s \cdot d\mathbf{l}$$

$$\mathbf{J}_s = n_s e^* \mathbf{v}_s$$

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

From London equations $\Phi' = 0$

for path which encloses no hole but only superconductors

$$\Phi' = \frac{1}{e^*} \oint \left(m^* \mathbf{v}_s + e^* \mathbf{A} \right) \cdot d\mathbf{l} = \frac{1}{e^*} \oint \mathbf{p} \cdot d\mathbf{l}$$

$$= \frac{nh}{e^*} = n\Phi_0$$

Sommerfeld quantum condition $\oint \mathbf{p} \cdot d\mathbf{l} = nh$

For Cooper-pair

$$e^* = 2e$$

$$\Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-7}\,\mathrm{Gs\ cm}^2 = 2.07 \times 10^{-15}\,\mathrm{Wb}$$

Single value of wave function

$$\mathbf{J} = \frac{e^*}{m^*}\left|\Psi\right|^2 \left(\hbar\nabla\varphi - e^*\mathbf{A}\right)$$

$$\frac{\hbar}{e^*}\nabla\varphi - \mathbf{A} = \mu_0\lambda^2\mathbf{J}$$

Take the line integral around a closed loop

$$\frac{\hbar}{e^*}\Delta\varphi = \mu_0\lambda^2\oint\mathbf{J}\cdot d\mathbf{l} + \oint\mathbf{A}\cdot d\mathbf{l}$$

$$\Delta\varphi = 2n\pi$$

→

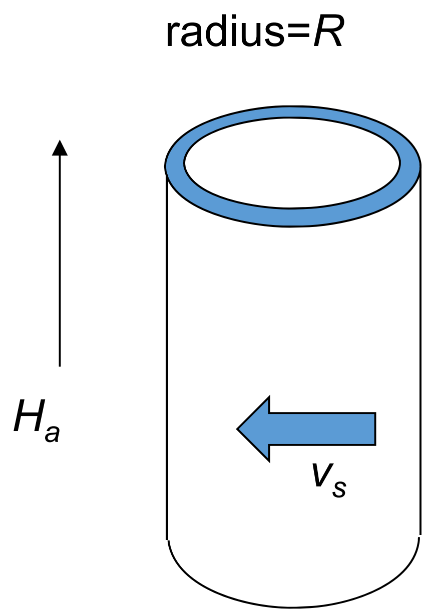
$$\oint\mathbf{A}\cdot d\mathbf{l} + \mu_0\lambda^2\oint\mathbf{J}\cdot d\mathbf{l} = \frac{nh}{e^*}$$

The Little-Park experiment

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \pi R^2 \mu_0 H_a$$

$$\Phi' = n\Phi_0 = \frac{m^*}{e^*} 2\pi R v_s + \Phi$$

$$v_s = \frac{\hbar}{m^* R}\left(n - \frac{\Phi}{\Phi_0} \right)$$



GL 1st eq.

$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A} \right)^2 \Psi = 0$$

$$\alpha + \beta|\Psi|^2 + \frac{m^* v_s^2}{2} = 0$$

If v_s is constant of **r**

$$|\Psi|^2 = \Psi_\infty^2 \left(1 - \frac{m^* v_s^2}{2|\alpha|} \right) = \Psi_\infty^2 \left[1 - \left(\frac{\xi m^* v_s}{\hbar} \right)^2 \right]$$

Near critical point $T \sim T_C(H)$ $\left|\Psi\right| \sim 0$

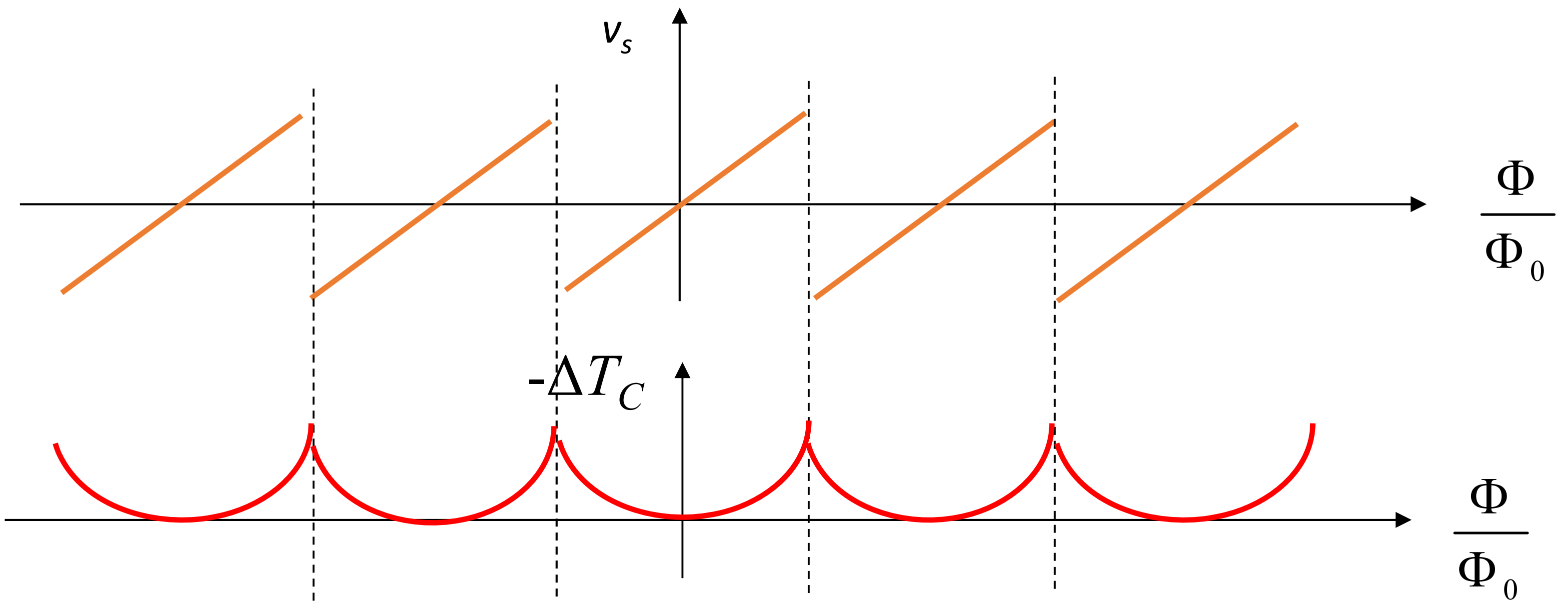
$$\frac{m^* v_s^2}{2\left|\alpha_c\right|} \approx 1 \qquad \alpha_C\left(H\right) \approx -\frac{1}{2} m^* v_s^2 \neq 0$$

$$\frac{1}{\xi\left(T\right)^2} = \frac{1}{R^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2$$

Define $t_H = \dfrac{T_C\left(H\right)}{T_C}$ $\alpha \propto \left(t - 1\right)$

$$\left(t_H - 1\right) \propto \alpha_c\left(H\right) \propto -v_s^2 \propto -\frac{1}{R^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2$$

$\Rightarrow$ $$\frac{\Delta T_C}{T_C} \propto \frac{1}{R^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2$$

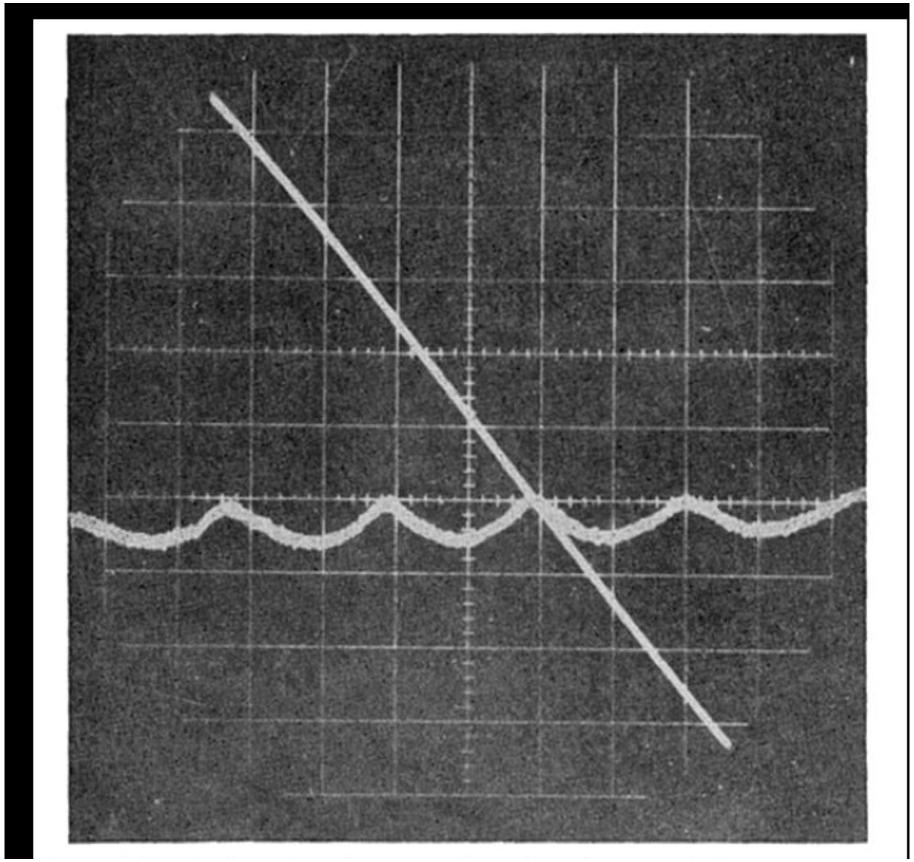


$$\frac{\xi(T)}{\xi_0} = \frac{1}{\sqrt{1-t}} \qquad 1-t \propto \frac{\xi_0^2}{\xi^2(T)} = \frac{\xi_0^2}{R^2}\left(n-\frac{\Phi}{\Phi_0}\right)^2$$

$$\frac{\Delta T_C}{T_C} = 0.55\left(\frac{\xi_0}{R}\right)^2\left(n-\frac{\Phi}{\Phi_0}\right)^2$$ Clean sample

$$\frac{\Delta T_C}{T_C} = 0.73\frac{\xi_0 l}{R^2}\left(n-\frac{\Phi}{\Phi_0}\right)^2$$ dirty sample

Experiment data



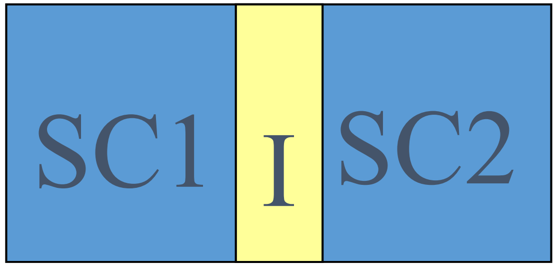
Resistance change near T_c

Parks, R. D. and W. A. Little (1964). "Fluxoid Quantization in a Multiply-Connected Superconductor." Physical Review **133**(1A): A97-A103.

Josephson tunneling

The superconducting leads can be expressed with a macroscopic wave function

Ψ_1 and Ψ_2



Wave function Ψ_1 Ψ_2

The wave equations

Eigenenergies for SC1 and SC2

$$i\hbar\frac{\partial\Psi_1}{\partial t} = U_1\Psi_1 + K\Psi_2$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = U_2\Psi_2 + K\Psi_1$$

K is the coupling energy between the two wave functions

$$\Psi_1 = |\Psi_1| e^{i\theta_1} \qquad \Psi_2 = |\Psi_2| e^{i\theta_2}$$

$$i\hbar\left(\frac{\partial|\Psi_1|}{\partial t} + i|\Psi_1|\frac{\partial\theta_1}{\partial t}\right) = U_1|\Psi_1| + K|\Psi_2|e^{i(\theta_2-\theta_1)}$$

$$i\hbar\left(\frac{\partial|\Psi_2|}{\partial t} + i|\Psi_2|\frac{\partial\theta_2}{\partial t}\right) = U_2|\Psi_2| + K|\Psi_1|e^{i(\theta_1-\theta_2)}$$

Real part: $$-\hbar|\Psi_1|\frac{\partial\theta_1}{\partial t} = U_1|\Psi_1| + K|\Psi_2|\cos(\theta_2-\theta_1)$$

→ Phase change

Imaginary part: $$\hbar\frac{\partial|\Psi_1|}{\partial t} = K|\Psi_2|\sin(\theta_2-\theta_1)$$

→ Particle number change

Josephson voltage-phase relation

$$-\frac{\partial \theta_1}{\partial t} = \frac{U_1}{\hbar} + \frac{K}{\hbar}\frac{|\Psi_2|}{|\Psi_1|}\cos(\theta_2 - \theta_1)$$

$$-\frac{\partial \theta_2}{\partial t} = \frac{U_2}{\hbar} + \frac{K}{\hbar}\frac{|\Psi_1|}{|\Psi_2|}\cos(\theta_1 - \theta_2)$$

with $|\Psi_2| \approx |\Psi_1|$

$$\phi = \theta_1 - \theta_2$$

Phase difference $\frac{\partial \phi}{\partial t} = \frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t} = \frac{U_2 - U_1}{\hbar} = \frac{2eV}{\hbar}$

(e^*=-2e)

$$\frac{\partial \phi}{\partial t} = \frac{2eV}{\hbar}$$

Josephson current-phase relation

$$\frac{\partial |\Psi_1|^2}{\partial t} = 2|\Psi_1|\frac{\partial |\Psi_1|}{\partial t} = \frac{2K}{\hbar}|\Psi_1||\Psi_2|\sin(\theta_2 - \theta_1)$$

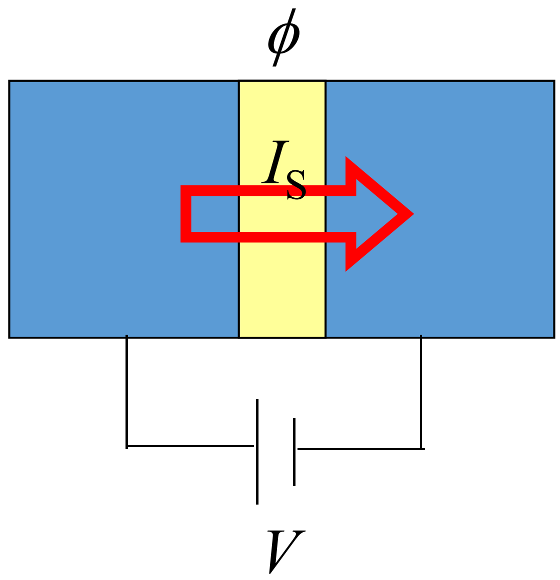
$$\frac{\partial |\Psi_2|^2}{\partial t} = 2|\Psi_2|\frac{\partial |\Psi_2|}{\partial t} = -\frac{2K}{\hbar}|\Psi_1||\Psi_2|\sin(\theta_2 - \theta_1)$$

Current= $I_S = -e^* \dfrac{\partial |\Psi_1|^2}{\partial t} = -\dfrac{2e^* K}{\hbar}|\Psi_1||\Psi_2|\sin\phi \equiv I_C \sin\phi$

$$I_S = I_C \sin\phi$$

Josephson energy

Starting from zero phase difference, the voltage source drives the junction to a phase difference= ϕ



The electrical power delivered= $I_S V$

Energy stored in juntion

=

$$\int I_S V dt = \int I_S \frac{\hbar}{2e}\frac{\partial \phi}{\partial t} dt \quad = \frac{\hbar}{2e}\int I_S d\phi = \frac{\hbar}{2e} I_C \int \sin\phi d\phi$$

$$= -E_J \cos\phi + \text{constant}$$

$$E_J = \frac{2e}{\hbar} I_C$$

The Josephson coupling energy

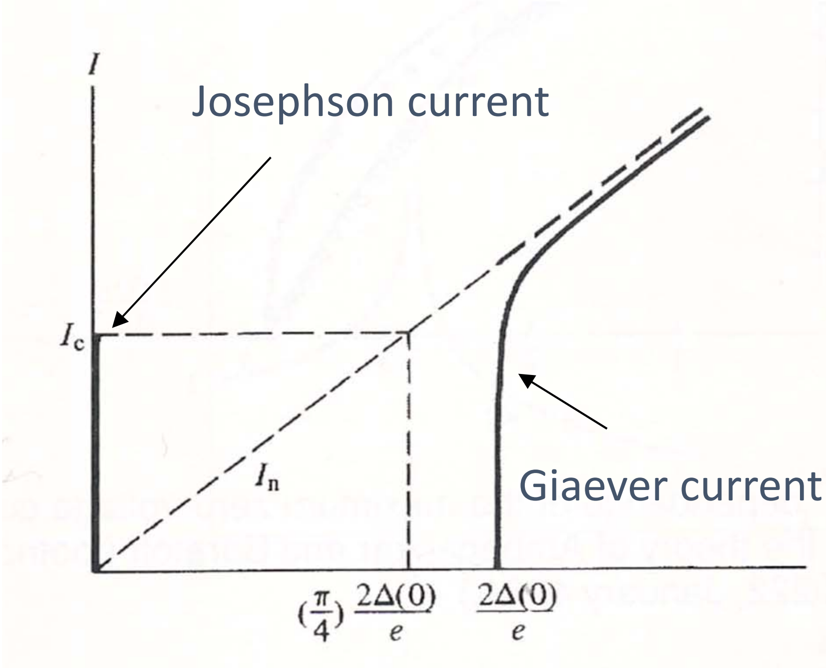
General expression of the critical current

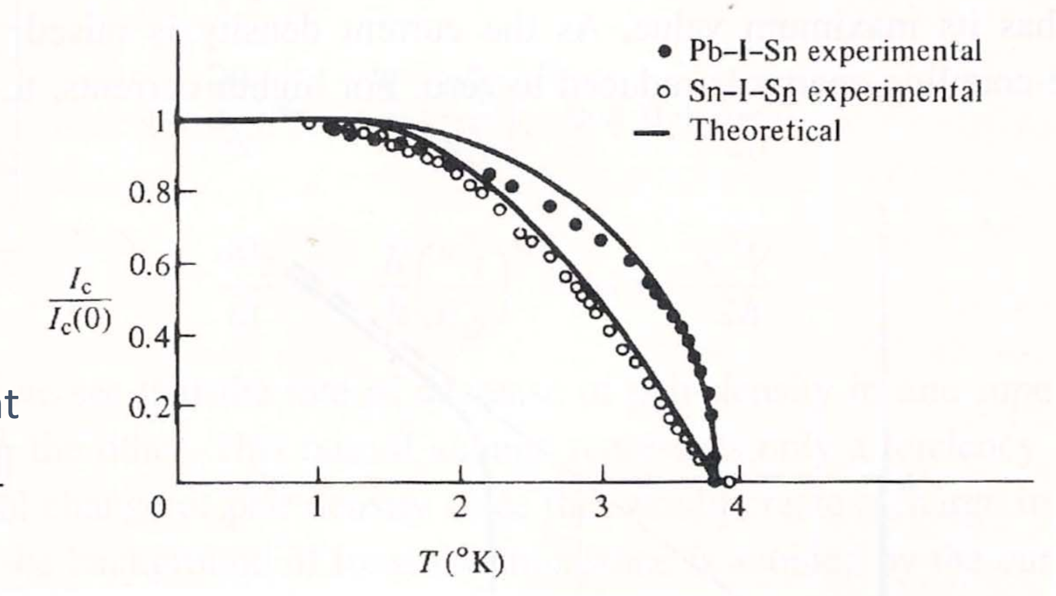
According to microscopic theory, the critical current of a Josephson junction is

$$I_C = G_n \frac{\pi\Delta(T)}{2e}\tanh\left(\frac{\Delta(T)}{2k_B T}\right)$$

Normal state conductance

For materials with strong electron-phonon coupling, the critical current is reduced





ac Josephson effect

$$I_S = I_C \sin\phi$$

$$\frac{\partial\phi}{\partial t} = \frac{2eV}{\hbar}$$

For V=0, ϕ=constant, current is constant of time

For $V \neq 0$, $\phi = \omega_0 t$

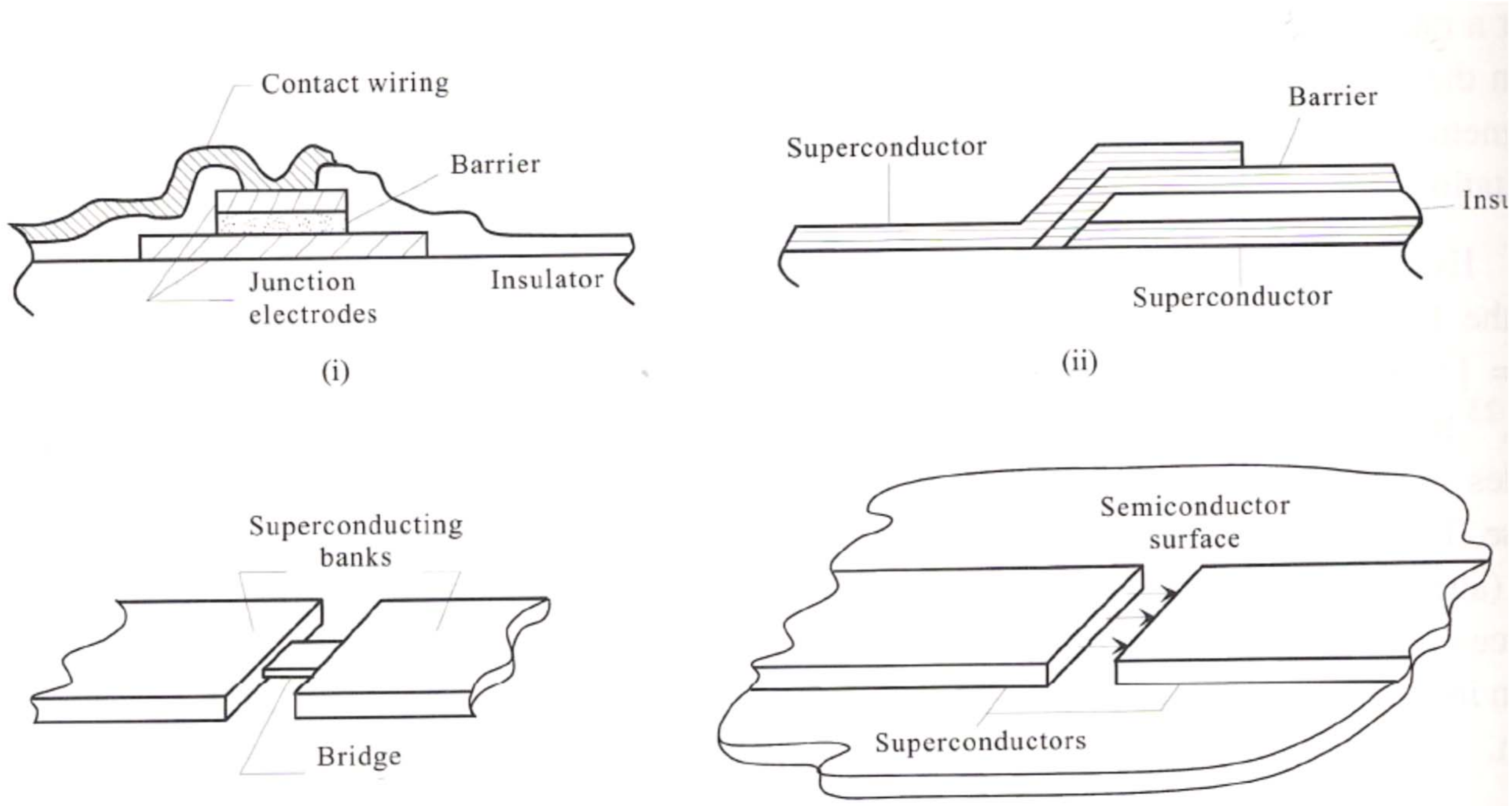
$$\omega_0 = \frac{2eV_0}{\hbar}$$

→ $$I_S = I_C \sin\left(\omega_0 t + \phi(0)\right)$$

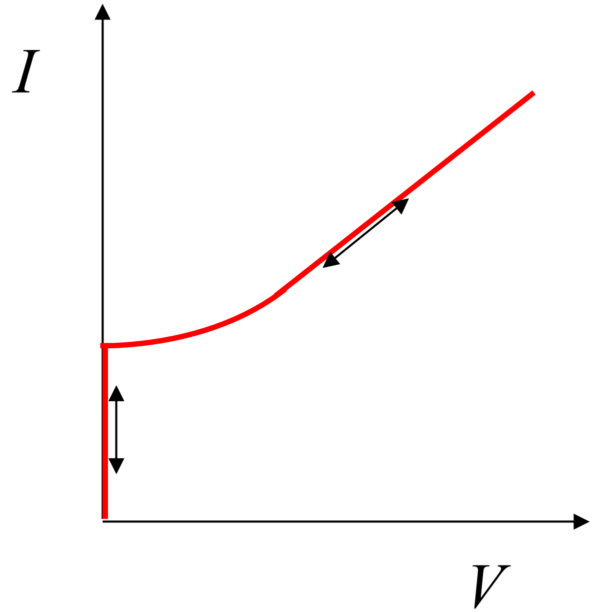
At finite DC voltage, the supercurrent is alternating with a angular frequency $\omega_0 = 2eV_0/\hbar$

$$f = 483.6\times10^{12}\,\mathrm{Hz/V}$$

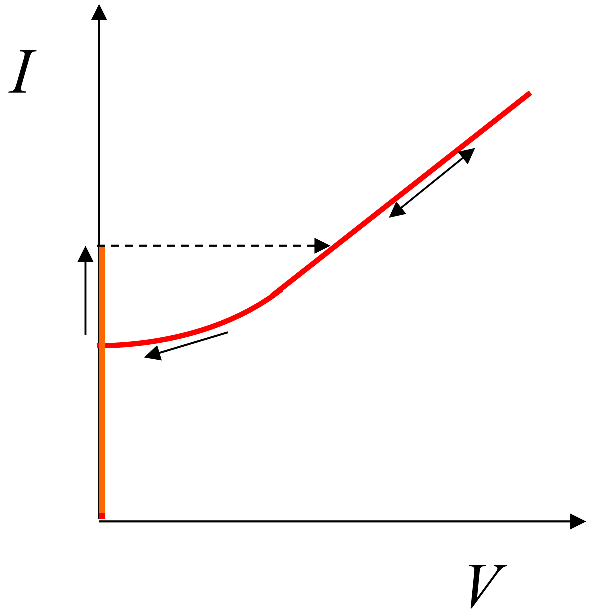
Conducting barrier junctions



Hysteretic junctions



Low resistance junctions: non-hysteretic behavior



high resistance junctions: hysteretic behavior

Josephson inductance

The josephson effect indicates a junction inductance:

$$I_S = I_C \sin\phi$$

$$\frac{\partial \phi}{\partial t} = \frac{2eV}{\hbar}$$

$$\frac{dI}{dt} = I_C \frac{d\phi}{dt}\cos\phi = \left(\frac{2eI_C}{\hbar}\cos\phi\right)V$$

$$= \frac{\cos\phi}{L_J}V$$

Josephson inductance $L_J = \frac{\hbar}{2eI_C}$

The dissipation would relax the motion of phase such that $(\phi \approx 0)$

$$\frac{dI}{dt} + \frac{I}{\tau_i} = \frac{1}{L_J}V$$

Linear damping

τ_i Relaxation time

The DC response should be $I = \frac{\tau_i}{L_J}V$ $\quad G_n = \frac{\tau_i}{L_J}$ $\quad \tau_i = G_n L_J$

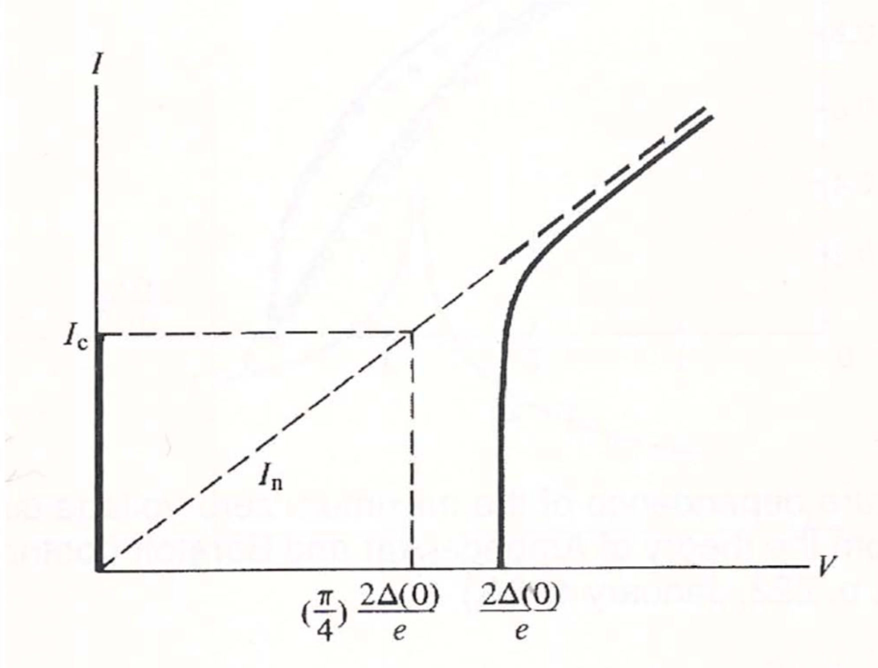
$I_C R_N$ product

$$\tau_i = G_n L_J = \frac{\hbar}{2eI_C R_n} \propto \frac{1}{I_C R_n}$$

At low temperatures

$$I_C \simeq G_n \frac{\pi\Delta(T)}{2e}$$

$$I_C R_n = \frac{\pi\Delta_0}{2e}$$

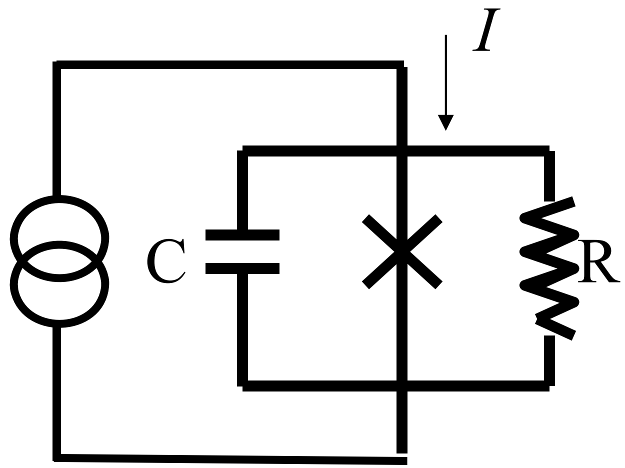


RCSJ model

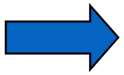
Resistively and capacitively shunted junction:

$$I = C\frac{dV}{dt} + \frac{V}{R} + i_C \sin\phi$$

capacitor resistor inductor



$$V = \frac{\hbar}{2e}\frac{d\phi}{dt}$$

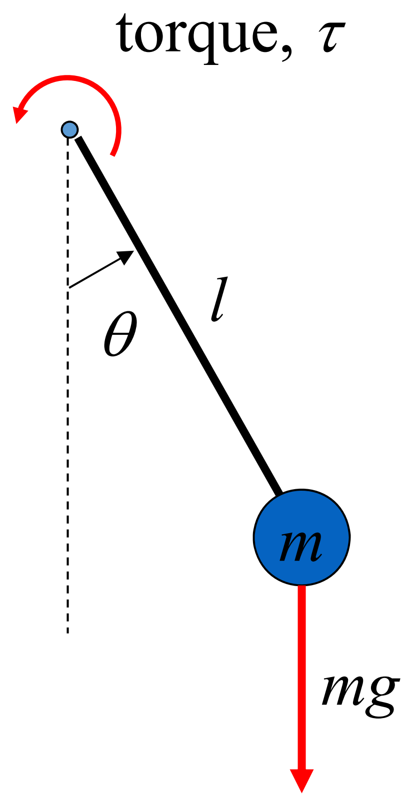


$$\frac{C\hbar}{2e}\frac{d^2\phi}{dt^2} + \frac{\hbar}{2eR}\frac{d\phi}{dt} + i_C \sin\phi = I$$

Kinetic term damping potential term Driving force

Pendulum analogy

RCSJ model	**pendulum**
ϕ	θ
I	τ
C	m
$1/R$	η
i_S	x
V	ω



The classical dynamics of a Josephson junction can be solved

Stewart-McCumber parameter

$$\frac{C\hbar}{2e}\frac{d^2\phi}{dt^2}+\frac{\hbar}{2eR}\frac{d\phi}{dt}+i_C\sin\phi=I$$

$$\Rightarrow\quad \frac{C\hbar}{2eI_Ce}\frac{d^2\phi}{dt^2}+\frac{\hbar}{2eI_CR}\frac{d\phi}{dt}+\sin\phi=\frac{I}{I_C}$$

$$\beta_c\frac{d^2\phi}{d\tau^2}+\frac{d\phi}{d\tau}+\sin\phi=\frac{I}{I_C}$$

$$L_J=\frac{\hbar}{2eI_C}$$

$$\tau=\frac{t}{\tau_J}$$

$$\beta_c=\frac{\dfrac{C\hbar}{2I_Ce}}{\left(\dfrac{L_J}{R}\right)^2}=\frac{R^2C}{L_J}=\frac{\tau_{RC}}{\tau_J}$$

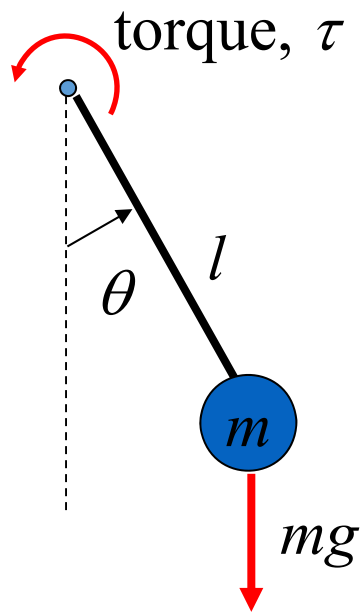
$$\tau_J=\frac{L_J}{R}$$

Josephson time constant

Stewart-McCumber parameter

Overdamped junctions

$\beta_c \ll 1$ Overdamped junctions



$$\frac{d\phi}{d\tau} + \sin\phi = \frac{I}{I_C}$$

Static solution $I < I_C$

$$\phi = \sin^{-1}\frac{I}{I_C}$$

dynamic solution $I > I_C$

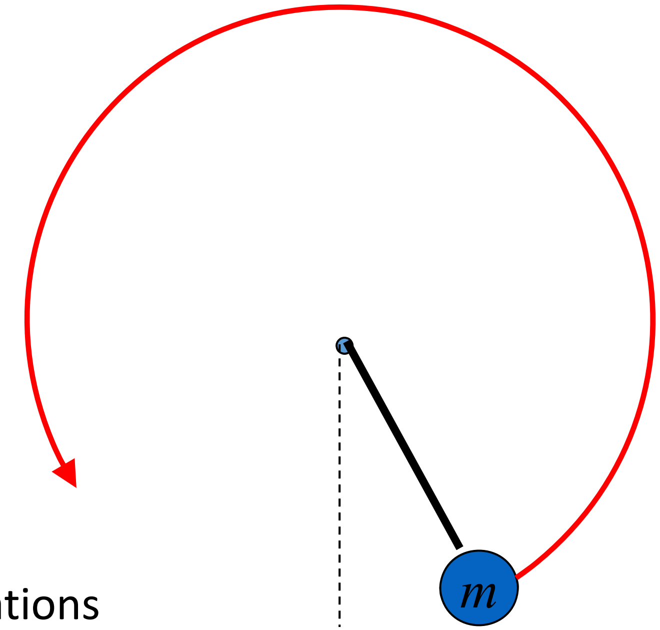
$$\frac{d\phi}{d\tau} = \frac{I}{I_C} - \sin\phi \quad \Rightarrow \quad \frac{d\phi}{\frac{I}{I_C} - \sin\phi} = d\tau$$

$$\phi(t) = 2\tan^{-1}\left[\sqrt{1-\left(\frac{I_C}{I}\right)^2}\tan\left(\frac{t\sqrt{(I/I_C)^2-1}}{2\tau_J}\right) - \frac{I_C}{I}\right]$$

Overdamped junctions

The solution is a periodic function in time

$$\phi(t) = 2\tan^{-1}\left[\sqrt{1-\left(\frac{I_C}{I}\right)^2}\tan\left(\pi\frac{t}{T}\right)-\frac{I_C}{I}\right]$$



Period= $T = \dfrac{2\pi\tau_J}{\sqrt{(I/I_C)^2-1}}$

We may calculate the static voltage using Josephson relations

$$\langle V\rangle = \frac{\hbar}{2e}\left\langle\frac{\partial\phi}{\partial t}\right\rangle = \frac{\hbar}{2e}\frac{1}{T}\int_0^T\left(\frac{\partial\phi}{\partial t}\right)dt = \frac{\hbar}{2e}\frac{1}{T}\int_0^{2\pi}d\phi$$

$$= \frac{\hbar}{2e}\frac{\sqrt{(I/I_C)^2-1}}{\tau_J}$$

$$\Rightarrow \quad \langle V\rangle = \frac{I_C}{R}\sqrt{(I/I_C)^2-1}$$

non-hysteretic behavior

Underdamped junctions

$\beta_c \gg 1$ underdamped junctions

$$\beta_c \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin\phi = \frac{I}{I_C}$$

Static solution $I < I_C$

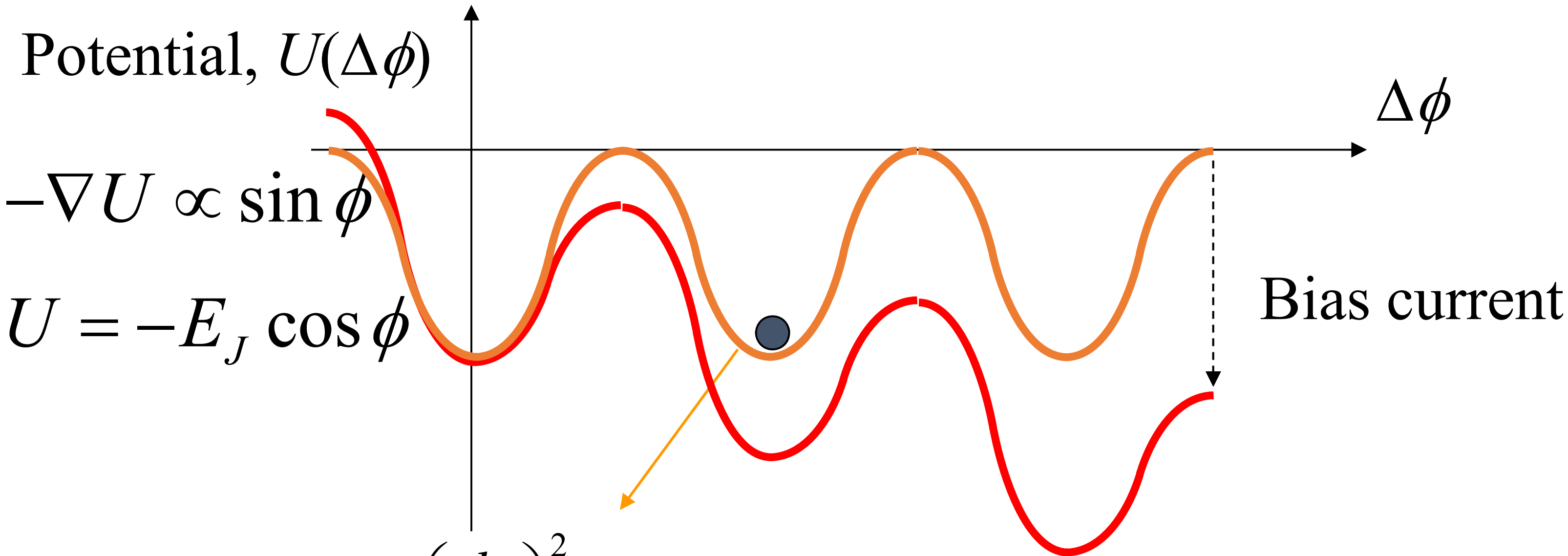
$$\phi = \sin^{-1}\frac{I}{I_C}$$

dynamic solution $I > I_C$

ϕ varies vary fast than RC time, so the static voltage comes from R and C

$$\langle V \rangle = IR$$

Tilted washboard model



Particle mass~ $\left(\frac{\hbar}{2e}\right)^2 C$

$$U = -E_J \cos\phi - \left(\frac{\hbar I}{2e}\right)\Delta\phi$$

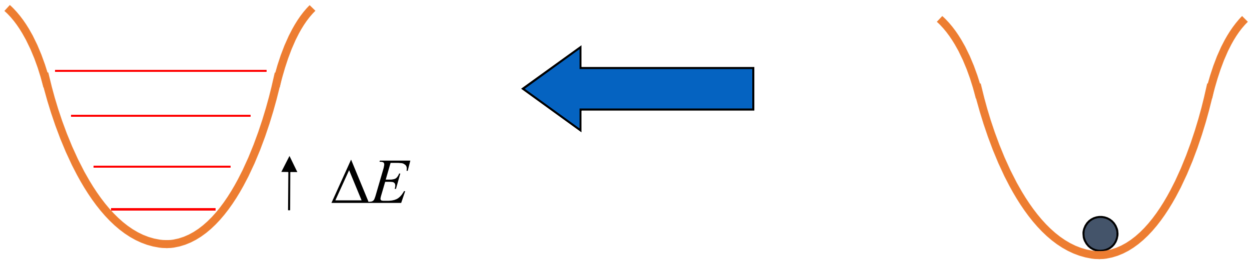
The particle is moving with a friction force proportional to its speed

$$\frac{1}{R}\left(\frac{\hbar}{2e}\right)^2 \frac{d}{dt}\Delta\phi$$

Quantum or classical?

In what situation the classical picture is plausible?

Consider the quantum mechanical excitations in the system



1D quantum well

Classical motion around a local potential minimum

$$U = -E_J \cos\phi \approx \boxed{\tfrac{1}{2} E_J \phi^2} - E_J$$

Simple harmonic potential with a spring constant of E_J

Particle mass $\sim \left(\frac{\hbar}{2e}\right)^2 C$

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{\text{spring constant}}{\text{mass}}} = \hbar\sqrt{\frac{E_J}{(\hbar/2e)^2\, C}} = 2\sqrt{2E_C E_J}$$

Charging energy $\quad E_C \equiv \dfrac{e^2}{2C}$

If barrier height E_J is much larger than ΔE, the junction can be treated classically.

$$\Rightarrow \qquad E_J \gg E_C$$