Rotation group

Spatial Translations

- the unitary operator for a spatial translation a is

$$T(a) = e^{-iP\cdot a/\hbar}$$

- where *a* is a numerical 3-vector, and *P* is the total momentum operator for the system in question

$$\left[P_i, P_j\right] = 0$$

- Let x_n be the coordinate operator of particle *n*.

$$T^\dagger(a) x_n T(a) = x_n + a$$

- If $|\varphi>$ is any state, then

$$T|\varphi\rangle = |\varphi; a\rangle$$

Wave functions

- Understand how the wave functions change by *T*. Consider 1D for example

$$\varphi(x) = \langle x | \varphi \rangle \qquad \varphi'(x) = \langle x | \varphi; a \rangle = \langle x | T | \varphi \rangle$$

- Evaluate how the position eigenstates change by T

$$T | x \rangle = | x; a \rangle \qquad [x, T(a)] = i \frac{\partial T(a)}{\partial p} = aT(a)$$

$$x | x; a \rangle = xT | x \rangle = Tx | x \rangle + [x, T] | x \rangle = (x + a)T | x \rangle = (x + a) | x; a \rangle$$

$$| x; a \rangle = | x + a \rangle$$

we have

$$\varphi'(x) = \langle T^{\dagger} x | \varphi \rangle = \langle x - a | \varphi \rangle = \varphi(x - a)$$

Groups of translation op

- Take the translation through *a* followed by *b*:

$$T(b)T(a) = T(a+b)$$

- the order in these translations does not matter; they commute.

- The special case *b* = -*a*,

$$T(a)T(-a) = T(a)T^{\dagger}(a) = 1$$

- the operators *T*(*a*) form an Abelian Lie group of unitary operators standing in one-to-one correspondence with the group of translation in the Euclidean 3-space E_3.

- A group (G) is a finite or infinite set of elements (g_1, g_2, . . .) having a composition law for every pair of elements such that $g_1 g_2$ is again an element of (G); which is associative, i.e., $(g_1 g_2)g_3 = g_1(g_2 g_3)$; and with every element g_i having an inverse g_i^{-1} such that $g_i g_i^{-1}$ is the identity element *I*, i.e., $Ig_i = g_i I = g_i$ for all *i*.

- A group is Abelian if all its elements commute, i.e., $g_1 g_2 = g_2 g_1$

- A group with an infinite set of elements is a Lie group if its elements can be uniquely specified by a set of continuous parameters (z_1 . . . z_r)

infinitesimal transformation

- the generalization of the infinitesimal translation

$$T(\delta a) = 1 - \frac{i}{\hbar}\delta a \cdot P$$

- if a unitary operator $U(z_l \ldots z_r)$ carries out a transformation belonging to a Lie group, then if the transformation is infinitesimal it has the form

$$U = 1 - i \sum_l \delta z_l \cdot \mathcal{G}_l$$

Generators

- the operators $\mathscr{G}_l$, which must be Hermitian for *U* to be unitary, are called the generators of the group (G).

- let $f(x_1, x_2, x_3)$ be any function of the coordinates in E_3, taken now to be real numbers and not operators, and consider the infinitesimal translation $x_i \to x_i + \delta a_i$

$$\delta f = f(x_i + \delta a_i) - f(x_i) = \sum_i \delta a_i \frac{\partial f}{\partial x_i}$$

$$\delta f = \frac{i}{\hbar} \sum_i \delta a_i \frac{\hbar}{i} \frac{\partial f}{\partial x_i}$$

Rotations

- Parametrization: specify a rotation *R* by the unit vector *n* along an axis of rotation, and an angle of rotation (θ) about that axis
- infinitesimal rotation will be parametrized by

$$n\delta\theta$$

- Under this rotation, a vector K in E_3 transforms as follows:

$$K \to K + \delta K = K + \delta\theta(n \times K)$$

$$\delta K = \delta\theta \epsilon_{ijk} n_j K_k$$

ε_{ijk} antisymmetric Levi-Civita tensor

Rotation group

- a unitary transformation $D(R)$ on the Hilbert space S) of the system of interest.

$$|\psi\rangle \longrightarrow |\psi'\rangle = D(R)|\psi\rangle$$

- For infinitesimal rotations

$$D^\dagger(R) r D(R) = r + \delta r = r + \delta\theta(n \times r)$$

$$\psi'(r) = \langle r|\psi'\rangle = \langle D^\dagger(R) r|\psi\rangle = \psi(r - \delta r)$$

Generator for rotation

- Consider an infinitesimal rotation about *n* = (0,0,1), the change in ψ is

$$\delta\psi(r) = \psi(r - \delta r) - \psi(r)$$

$$\delta r = \delta\theta \hat{z} \times r = (y, -x, 0)\delta\theta$$

$$\delta\psi(r) = \delta\theta \left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y} \right) = \frac{i}{\hbar}\delta\theta(xp_y - yp_x)\psi$$

$$= \frac{i}{\hbar}\delta\theta L_z\psi$$

- The rotation generator is angular momentum

Angular momentum

- The general rotation can be expressed as

$$D(R) = \exp\left(-\frac{i}{\hbar}\theta n \cdot J\right)$$

- *n.J* is the component of angular momentum along the direction *n*.

non-Abelian group

- Successive rotations of K about distinct axes do not commute, a fact that is captured in the commutation rule

$$\left[J_i, J_j\right] = i\epsilon_{ijk}J_k$$

- The rotation group is non-Abelian

$$D(R_2)D(R_1) \neq D(R_1)D(R_2)$$

Dimensionless angular momentum

- Consider a single particle with position and momentum operators *x* and *p*. The (dimensionless) orbital angular momentum operator *L* for this particle is then defined as

$$L = \frac{1}{\hbar}(x \times p) \qquad L_i = \frac{1}{\hbar}\epsilon_{ijk} x_j p_k$$

- the order of x_j and p_k does not matter because only commuting factors appear

$$\left[x_j, p_k\right] = i\hbar\delta_{jk}$$

- The commutation rule for the orbital angular momentum

$$\left[L_i, L_j\right] = \frac{1}{\hbar^2}\left[\epsilon_{ikl} x_k p_l, \epsilon_{jmn} x_m p_n\right] = \frac{\epsilon_{ikl}\epsilon_{jmn}}{\hbar^2}\left[x_k p_l, x_m p_n\right]$$

$$\begin{aligned}\left[x_k p_l, x_m p_n\right] &= \left[x_k, x_m p_n\right] p_l + x_k \left[p_l, x_m p_n\right] \\ &= x_m \left[x_k, p_n\right] p_l + x_k \left[p_l, x_m\right] p_n \\ &= i\hbar\left(\delta_{kn} x_m p_l - \delta_{lm} x_k p_n\right)\end{aligned}$$

$$
\begin{aligned}
\left[L_i, L_j\right] &= \frac{i}{\hbar}\left(\epsilon_{ikl}\epsilon_{jmk}x_m p_l - \epsilon_{ikl}\epsilon_{jln}x_k p_n\right) \\
&= \frac{i}{\hbar}\left(\epsilon_{kli}\epsilon_{kjm}x_m p_l - \epsilon_{lik}\epsilon_{lnj}x_k p_n\right) \\
&= \frac{i}{\hbar}\left[\left(\delta_{jl}\delta_{im} - \delta_{ij}\delta_{lm}\right)x_m p_l - \left(\delta_{in}\delta_{jk} - \delta_{ij}\delta_{kn}\right)x_k p_n\right] \\
&= \frac{i}{\hbar}\left[\left(x_i p_j - x_j p_i\right) - \delta_{ij}\left(x_l p_l - x_k p_k\right)\right] \\
&= \frac{i}{\hbar}\left(x_i p_j - x_j p_i\right) \\
&= i\epsilon_{ijk}L_k
\end{aligned}
$$

Here we used the identity $\epsilon_{ikl}\epsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$