Spin

spin 1/2 system

- A particle may have an intrinsic angular momentum called spin
- Electrons, protons, and neutrons are all examples of spin-1/2 particles
- If one measure the z-component S_z(or S_x, S_y) of the spin angular momentum for one of these particles, he gets

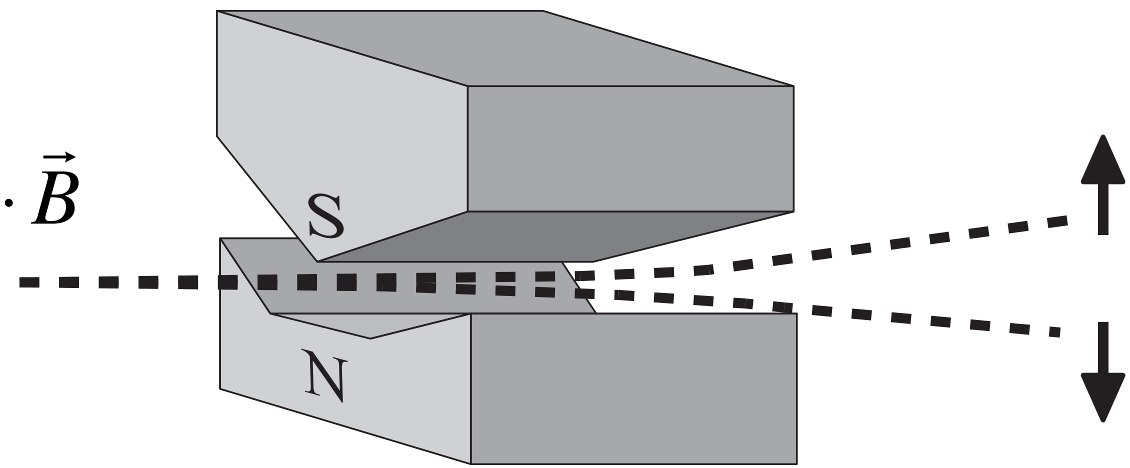
$$S_z = \pm \frac{\hbar}{2}$$

Stern-Gerlach experiment

- A stream of atoms moving from the right passes between the asymmetric poles of a magnet. Particles with different values of μ_z are deflected in different directions. The final position of the atom determines its μ_z

$\vec{\mu} = \gamma \vec{S}$ γ is gyromagnetic ratio

$E = -\vec{\mu} \cdot \vec{B}$



the spin state

- superpositions of spin-up and spin-down states

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|z_+\rangle + \beta|z_-\rangle$$

Bloch sphere

$$|x_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{1}{\sqrt{2}}|z_-\rangle$$

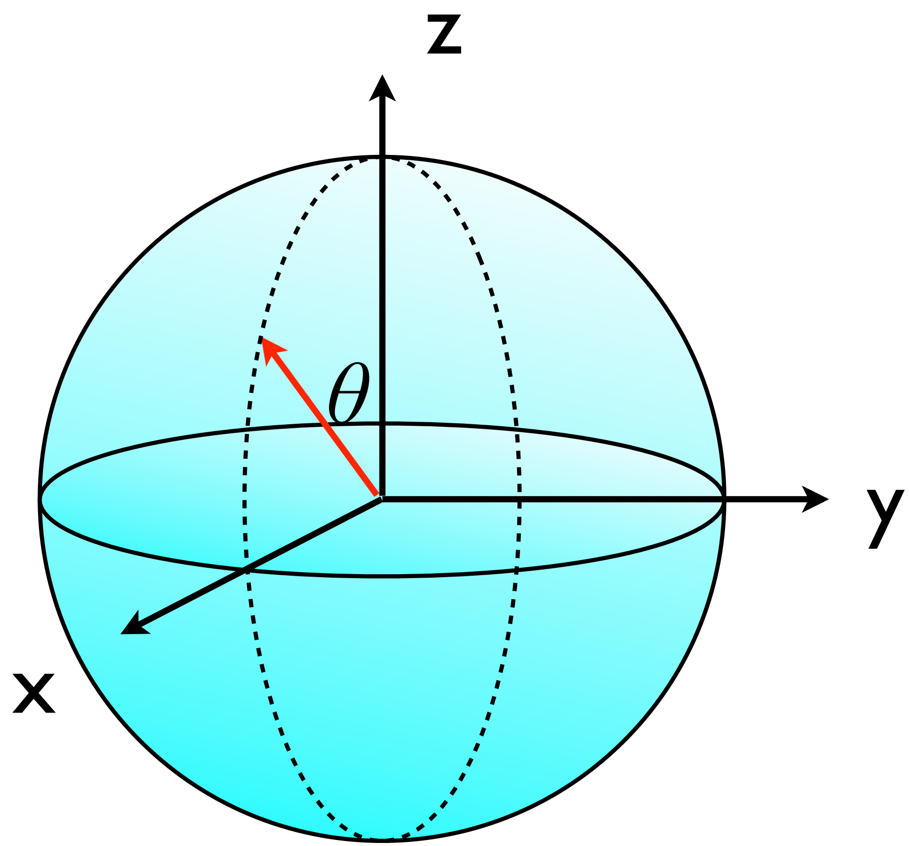
$$|x_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle$$

why? $\langle x_- | x_+ \rangle = 0$

$$|\langle z_+ | x_+ \rangle|^2 = |\langle z_- | x_+ \rangle|^2 = \frac{1}{2}$$

$$|y_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{i}{\sqrt{2}}|z_-\rangle$$

$$|y_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{i}{\sqrt{2}}|z_-\rangle$$



change of basis

- Suppose we choose a direction in the xz-plane that is inclined at an angle θ from the z-axis. Then the amplitude vectors

$$\left|\theta_+\right\rangle = \cos\frac{\theta}{2}\left|z_+\right\rangle + \sin\frac{\theta}{2}\left|z_-\right\rangle$$

$$\left|\theta_-\right\rangle = \sin\frac{\theta}{2}\left|z_+\right\rangle - \cos\frac{\theta}{2}\left|z_-\right\rangle$$

Pauli operators

- Hermitian operators in 2 level systems

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |0\rangle\langle 1|$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|0\rangle\langle 1| + i|0\rangle\langle 1|$$

$$S_\theta = \frac{\hbar}{2}\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right) = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Projection operator

- the projection to +x and -x direction

$$\left|x_+\right\rangle\left\langle x_+\right| = \left(\frac{1}{\sqrt{2}}\left|z_+\right\rangle + \frac{1}{\sqrt{2}}\left|z_-\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left\langle z_+\right| + \frac{1}{\sqrt{2}}\left\langle z_-\right|\right)$$

$$= \frac{1}{2}\left(\left|z_+\right\rangle\left\langle z_+\right| + \left|z_-\right\rangle\left\langle z_-\right| + \left|z_+\right\rangle\left\langle z_-\right| + \left|z_-\right\rangle\left\langle z_+\right|\right)$$

$$\left|x_+\right\rangle\left\langle x_+\right| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\left|x_-\right\rangle\left\langle x_-\right| = \left(\frac{1}{\sqrt{2}}\left|z_+\right\rangle - \frac{1}{\sqrt{2}}\left|z_-\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left\langle z_+\right| - \frac{1}{\sqrt{2}}\left\langle z_-\right|\right)$$

$$= \frac{1}{2}\left(\left|z_+\right\rangle\left\langle z_+\right| + \left|z_-\right\rangle\left\langle z_-\right| - \left|z_+\right\rangle\left\langle z_-\right| - \left|z_-\right\rangle\left\langle z_+\right|\right)$$

$$\left|x_-\right\rangle\left\langle x_-\right| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P_{x\pm}^2 = P_{x\pm}$$

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\left(\left|x_+\right\rangle\left\langle x_+\right| - \left|x_-\right\rangle\left\langle x_-\right|\right) = \frac{\hbar}{2}\left(\left|z_+\right\rangle\left\langle z_-\right| + \left|z_-\right\rangle\left\langle z_+\right|\right)$$

$$\left\langle S_x\right\rangle = \frac{\hbar}{2}\left\langle\psi\right|\sigma_x\left|\psi\right\rangle$$

eigenvectors

- the eigenvectors of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \alpha = \pm\beta \qquad |x_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

- the eigenvectors of S$_\theta$

$$S_\theta = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \lambda^2 - \cos^2\theta - \sin^2\theta = 0 \qquad \lambda = \pm 1$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |\theta_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad |\theta_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

commutation relations

- the products of Pauli matrices

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z$$

$$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x \qquad \sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$$

- The commutators

$$\left[\sigma_a, \sigma_b\right] = 2i\varepsilon_{abc}\sigma_c \qquad \left[\sigma_x, \sigma_y\right] = 2i\sigma_z$$

- the anti-commutator

$$\left\{\sigma_a, \sigma_b\right\} = 2\delta_{ab}$$

S^2

- The length of spin vector

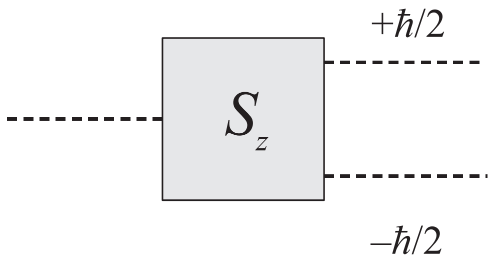
$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) = \frac{3\hbar^2}{4}$$

- S^2 is the identity operator multiplied by a constant. Any spin state has a definite S^2 value

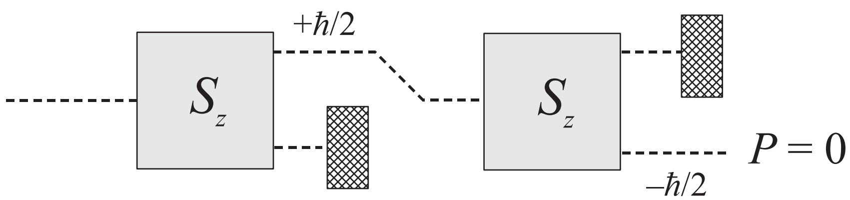
$$\left[S^2, S_i\right] = 0$$

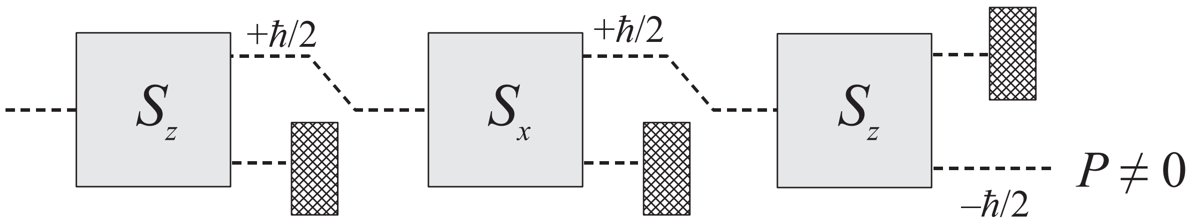
spin filters

Stern–Gerlach apparatus



Stern–Gerlach filters

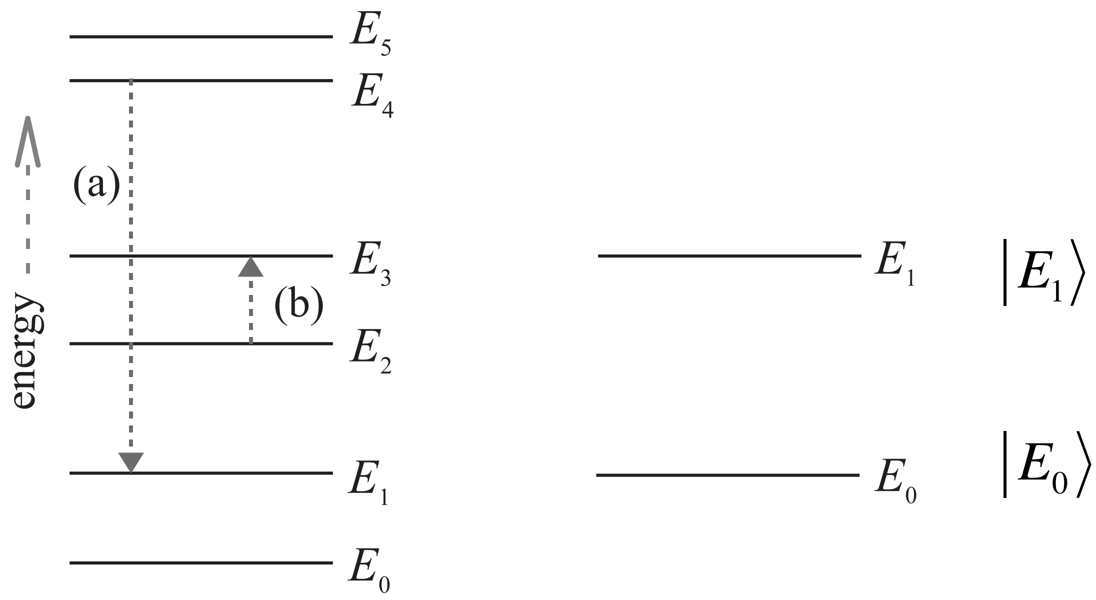




- S_z and S_x are complementary quantities

Energy levels and quantum states

- An atom generally has many different energy levels. In many experiments only two energy levels – usually the ground state and one excited state – play any significant role. In this case, we can adopt a simplified model, the two-level atom,



Time evolution

- In general, then, the atom will be in a state

$$|\psi\rangle = \alpha|E_0\rangle + \beta|E_1\rangle$$

- at t = 0 the state is $|\psi(0)\rangle = |E_k\rangle$, then at a later time

$$|\psi(t)\rangle = e^{-i\omega_k t}|E_k\rangle \qquad E_k = \hbar\omega_k$$

- probability P_u at time t

$$P_u(t) = |\langle u|\psi(t)\rangle|^2 = |\langle u|\psi(t)\rangle|^2 = P_u$$

stationary states

time evolution

$$|\psi\rangle = \alpha|E_0\rangle + \beta|E_1\rangle \qquad |\psi(t)\rangle = \alpha e^{-i\omega_0 t}|E_0\rangle + \beta e^{-i\omega_1 t}|E_1\rangle$$

- the relative phases of the two terms will change

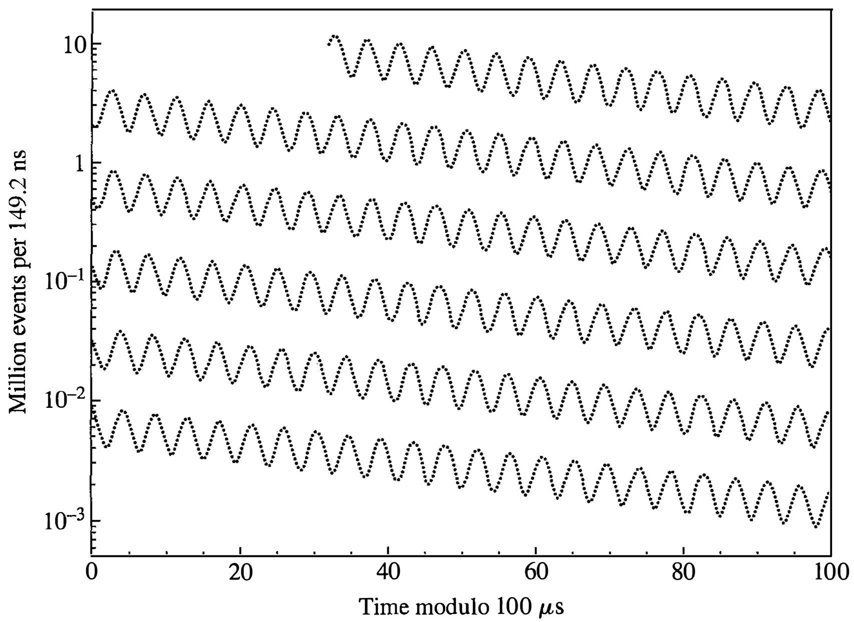
$$|\psi(0)\rangle = |u\rangle = \frac{1}{\sqrt{2}}|E_0\rangle + \frac{1}{\sqrt{2}}|E_1\rangle \qquad |\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t}|E_0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega_1 t}|E_1\rangle$$

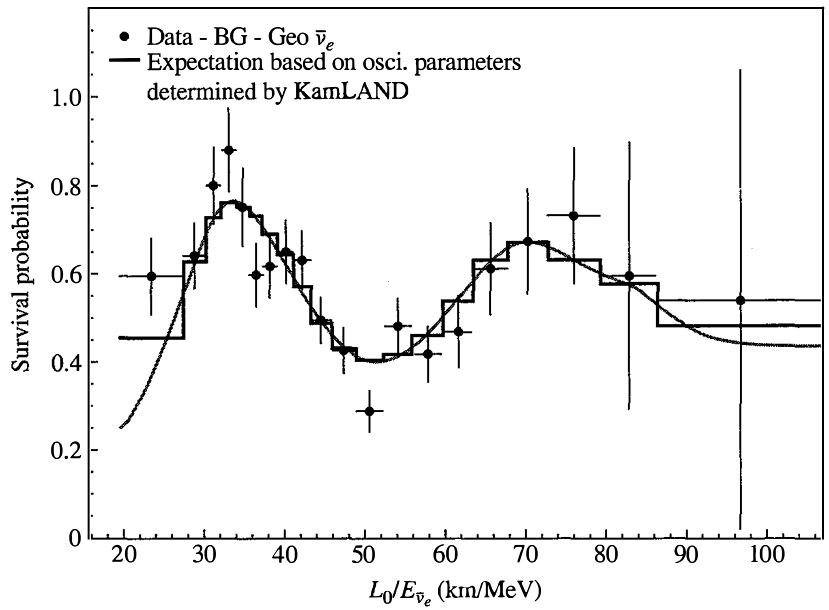
$$\langle u|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega_0 t} + e^{-i\omega_1 t}\right)$$

$$P_u(t) = \left|\langle u|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|e^{-i\omega_0 t} + e^{-i\omega_1 t}\right|^2 = \frac{1}{2}\left(1 + \cos\Delta\omega_0 t\right)$$

- As time progresses, the probability $P_u(t)$ of the measurement outcome u changes from 1 to 0 and then back to 1 again with an angular frequency $\Delta\omega = \omega_1 - \omega_0$

- Precession of muon spin PRD73, 072003(2006)





- Neutrino oscillation PRL100, 221803 (2008)

time evolution operator

- $U(t)|E_k\rangle = e^{-i\omega_k t}|E_k\rangle$ for an energy level state $|E_k\rangle$
- U(t) acts on states in a linear way

$$U(t)|\psi(0)\rangle = |\psi(t)\rangle$$

- The product of time evolution operators

$$U(t_2) = U(t_2 - t_1)U(t_1)$$

Hamiltonian operator

- $H|E_k\rangle = E_k\,|E_k\rangle$ for an energy level state $|E_k\rangle$
- H acts on states in a linear way.

$$|\psi(t)\rangle = \alpha e^{-i\omega_0 t}|E_0\rangle + \beta e^{-i\omega_1 t}|E_1\rangle$$

$$\Rightarrow \quad i\hbar\frac{d}{dt}|\psi(t)\rangle = \alpha E_0 e^{-i\omega_0 t}|E_0\rangle + \beta E_1 e^{-i\omega_1 t}|E_1\rangle = H|\psi(t)\rangle$$

- Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

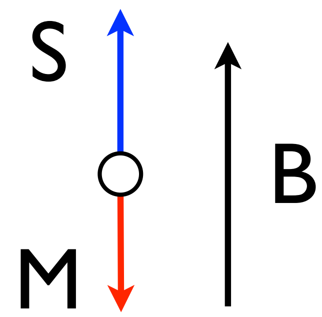
intrinsic magnetic moment

- electron has an intrinsic magnetic dipole moment by virtue of its spin

$$\mathbf{M} = -\frac{eg}{2m_e}\mathbf{S}$$

- gyromagnetic ratio, g=2
- Hamiltonian

$$H = -\mathbf{M}\cdot\mathbf{B} = \frac{eg\hbar}{4m_e}\sigma\cdot\mathbf{B}$$



ground state

Schrodinger equation

- Schrodinger equation $$i\hbar\frac{d\psi}{dt} = H\psi = \frac{eg\hbar}{4m_e}\sigma\cdot\mathbf{B}\psi$$
- If B in z-direction $$i\hbar\frac{d\psi}{dt} = \frac{eg\hbar}{4m_e}\sigma_z\psi$$
- the spinor state $$\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$$
- for the energy eigenstate $$\psi(t) = e^{-i\omega t}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

eigenstate

- eigen equation $$\frac{eg}{4m_e}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$
- eigenstates $$\omega = \pm\frac{eg}{4m_e} = \pm\omega_0$$ $$\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ $$\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- general solution

$$\psi(t) = ae^{-i\omega_0 t}\phi_+ + be^{i\omega_0 t}\phi_- = \begin{pmatrix} ae^{-i\omega_0 t} \\ be^{i\omega_0 t} \end{pmatrix}$$

spin precession

$$\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \qquad |u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\phi}{2}} \\ e^{i\frac{\phi}{2}} \end{pmatrix}$$

- Set initial state to be in x-direction

$$\phi = 0 \qquad \psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

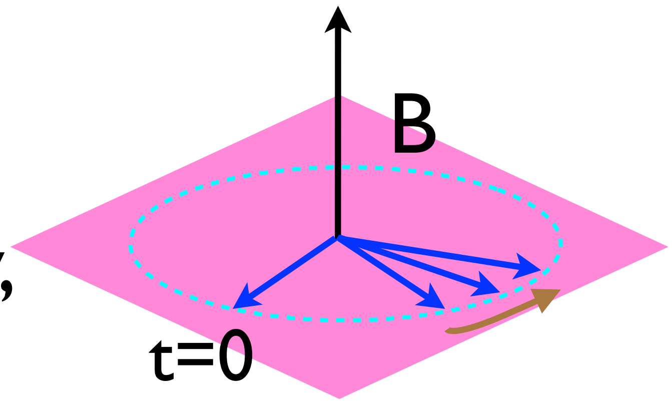
- for arbitrary time

$$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$$

- The expectation value

$$\langle S_x \rangle = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} e^{i\omega_0 t} & e^{-i\omega_0 t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{4}\left(e^{2i\omega_0 t} + e^{-2i\omega_0 t}\right) = \frac{\hbar\cos 2\omega_0 t}{2}$$

spin precession



- The spin precession frequency, called Larmor frequency

$$\Omega = 2\omega_0 = \frac{egB}{2m_e} = g\omega_c$$

- For B=1T, ω_c ~ 0.9 x 10^{11} rad/s

Paramagnetic resonance

- The magnetic field has a small oscillating part

$$\mathbf{B} = B_0\hat{z} + B_1\cos\omega t\hat{x}$$

- solve the Schrodinger equation

$$i\hbar\frac{d}{dt}\psi = \frac{eg\hbar}{4m_e}\begin{pmatrix} B_0 & B_1\cos\omega t \\ B_1\cos\omega t & -B_0 \end{pmatrix}\psi \qquad \psi = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$i\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{eg}{4m_e}\begin{pmatrix} B_0 & B_1\cos\omega t \\ B_1\cos\omega t & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

- When B_1=0 $\quad \psi_0 = \begin{pmatrix} a(0)e^{-i\omega_0 t} \\ b(0)e^{i\omega_0 t} \end{pmatrix}$

Paramagnetic resonance

- When B₁ <> 0, the solution $\psi \approx \psi_0$
- Slowly varying functions A and B

$$a(t)e^{i\omega_0 t} = A(t)$$

$$b(t)e^{-i\omega_0 t} = B(t)$$

- Consider how A and B evolve with time

$$i\frac{dA(t)}{dt} = i\frac{da(t)}{dt}e^{i\omega_0 t} - \omega_0 a(t)e^{i\omega_0 t} = \omega_0 a(t)e^{i\omega_0 t} + \omega_1 b(t)\cos(\omega t)e^{i\omega_0 t} - \omega_0 A(t)$$

$$= \omega_1 b(t)\cos(\omega t)e^{i\omega_0 t} = \omega_1 B(t)\cos(\omega t)e^{2i\omega_0 t} = \frac{1}{2}\omega_1 B(t)\left(e^{2i\omega_0 t + i\omega t} + e^{2i\omega_0 t - i\omega t}\right)$$

$$i\frac{dB(t)}{dt} = \frac{1}{2}\omega_1 A(t)\left(e^{-2i\omega_0 t + i\omega t} + e^{-2i\omega_0 t - i\omega t}\right) \qquad \omega_1 = \frac{egB_1}{4m_e}$$

Rotating wave approximation

- When the driving frequency is close resonance that

$$\omega \approx 2\omega_0$$

- There are rapid oscillating and slow oscillating terms
- The rotating wave approximation states that only slow oscillating term is important

$$\left(e^{\pm 2i\omega_0 t + i\omega t} + e^{\pm 2i\omega_0 t - i\omega t}\right) \simeq e^{\pm\left(2i\omega_0 t - i\omega t\right)}$$

Rabi oscillation

- To solve the coupled equation

$$i\frac{dA(t)}{dt} \approx \frac{1}{2}\omega_1 B(t) e^{2i\omega_0 t - i\omega t} \qquad i\frac{dB(t)}{dt} \approx \frac{1}{2}\omega_1 A(t) e^{-2i\omega_0 t + i\omega t}$$

$$\frac{d^2A(t)}{dt^2} \approx -\frac{i}{2}\omega_1 e^{2i\omega_0 t - i\omega t}\frac{dB(t)}{dt} + \frac{1}{2}\omega_1 (2\omega_0 - \omega) e^{2i\omega_0 t - i\omega t} B(t)$$

$$= \left(\frac{\omega_1}{2}\right)^2 A(t) + i(2\omega_0 - \omega)\frac{dA(t)}{dt}$$

- The solution is Rabi frequency

$$A(t) = A(0)e^{i\Omega t} \qquad -\Omega^2 = \left(\frac{\omega_1}{2}\right)^2 - (2\omega_0 - \omega)\Omega$$

$$\Omega = \left(\omega_0 - \frac{\omega}{2}\right) \pm \sqrt{\left(\omega_0 - \frac{\omega}{2}\right)^2 + \left(\frac{\omega_1}{2}\right)^2}$$

State evolution

- General solution $A(t) = A_+ e^{i\Omega_+ t} + A_- e^{i\Omega_- t}$

$$B(t) = e^{-2i\omega_0 t + i\omega t} \frac{2i}{\omega_1} \frac{dA(t)}{dt} = -\frac{2}{\omega_1} e^{-2i\omega_0 t + i\omega t} \left(A_+ \Omega_+ e^{i\Omega_+ t} + A_- \Omega_- e^{i\Omega_- t} \right)$$

$$= -\frac{2}{\omega_1} \left(A_+ \Omega_+ e^{-i\Omega_- t} + A_- \Omega_- e^{-i\Omega_+ t} \right)$$

- Suppose t=0

$$\psi = \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- The coefficients

$$A(0) = a(0) = 1 \qquad A_+ + A_- = 1 \qquad A_+ = \frac{\Omega_-}{\Omega_- - \Omega_+}$$

$$B(0) = b(0) = 0 \qquad A_+ \Omega_+ + A_- \Omega_- = 0 \qquad A_- = -\frac{\Omega_+}{\Omega_- - \Omega_+}$$

state evolution

- The probability to find the spin pointing in -z direction is

$$P_-(t) = |b(t)|^2 = |B(t)|^2 = \left(\frac{2}{\omega_1}\right)^2 \left|A_+\Omega_+ e^{-i\Omega_- t} + A_-\Omega_- e^{-i\Omega_+ t}\right|^2$$

$$= \left(\frac{2}{\omega_1}\right)^2 \left(\frac{\Omega_-\Omega_+}{\Omega_- - \Omega_+}\right)^2 \left|e^{-i\Omega_- t} - e^{-i\Omega_+ t}\right|^2$$

$$= 2\left(\frac{2}{\omega_1}\right)^2 \left(\frac{\Omega_-\Omega_+}{\Omega_- - \Omega_+}\right)^2 \left[1 - \cos(\Omega_- - \Omega_+)t\right]$$

$$= \frac{1}{2}\frac{\omega_1^2}{(2\omega_0 - \omega)^2 + \omega_1^2}\left[1 - \cos\sqrt{(2\omega_0 - \omega)^2 + \omega_1^2}\, t\right]$$

$$\Omega_+\Omega_- = -\left(\frac{\omega_1}{2}\right)^2$$

$$\Omega_+ + \Omega_- = 2\omega_0 - \omega$$

$$\Omega_+ - \Omega_- = \sqrt{(2\omega_0 - \omega)^2 + \omega_1^2}$$

resonance condition

- when $\omega = 2\omega_0$ $\Omega = \pm\frac{\omega_1}{2}$

- The down-spin probability $P_-(t) = \frac{1}{2}(1 - \cos\omega_1 t)$

- For nuclear spin $\omega_1 = \frac{egB_1}{4m_n}$

nuclear spin resonance

- a proton has a gyromagnetic ratio $\gamma_p = 2.675 \times 10^8\ s^{-1}T^{-1}$

- Larmor frequency at B=10T

$$\Omega = \gamma_p B = 2.675 \times 10^9\ s^{-1}$$

frequency =425.7 MHz

Nuclear magnetic resonance

Particle	Spin	w_{Larmor}/B $s^{-1}T^{-1}$	n/B
Electron	1/2	1.7608×10^{11}	28.025 GHz/T
Proton	1/2	2.6753×10^{8}	42.5781 MHz/T
Deuteron	1	0.4107×10^{8}	6.5357 MHz/T
Neutron	1/2	1.8326×10^{8}	29.1667 MHz/T
^{23}Na	3/2	0.7076×10^{8}	11.2618 MHz/T
^{31}P	1/2	1.0829×10^{8}	17.2349 MHz/T
^{14}N	1	0.1935×10^{8}	3.08 MHz/T
^{13}C	1/2	0.6729×10^{8}	10.71 MHz/T
^{19}F	1/2	2.518×10^{8}	40.08 MHz/T

900MHz, B=21.1 T

Addition of two spins

- The 2 spin system
- electron 1 $\left[S_{1x}, S_{1y}\right] = i\hbar S_{1z}$
- electron 2 $\left[S_{2x}, S_{2y}\right] = i\hbar S_{2z}$

$$\left[S_{1i}, S_{2j}\right] = 0 \qquad \text{for all } i, j$$

Total spin

- Total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$
- commutation relation

$$
\begin{aligned}
\left[S_x, S_y\right] &= \left[S_{1x} + S_{2x}, S_{1y} + S_{2y}\right] \\
&= \left[S_{1x}, S_{1y}\right] + \left[S_{2x}, S_{2y}\right] \\
&= i\hbar S_{1z} + i\hbar S_{2z} \\
&= i\hbar S_z
\end{aligned}
$$

- Therefor it is easy to find total spin S satisfies the commutation relation of an angular momentum

Eigenvalues

- Consider the states using single spinors
- electron 1 $\chi_{\pm}^{(1)}$

$$S_1^2 \chi_{\pm}^{(1)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_{\pm}^{(1)}$$

$$S_{1z} \chi_{\pm}^{(1)} = \pm\frac{1}{2}\hbar \chi_{\pm}^{(1)}$$

- electron 2 $\chi_{\pm}^{(2)}$

$$S_2^2 \chi_{\pm}^{(2)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_{\pm}^{(2)}$$

$$S_{2z} \chi_{\pm}^{(2)} = \pm\frac{1}{2}\hbar \chi_{\pm}^{(2)}$$

product states

- The possible states are (product states)

$$\chi_+^{(1)}\chi_+^{(2)} \qquad \chi_+^{(1)}\chi_-^{(2)} \qquad \chi_-^{(1)}\chi_+^{(2)} \qquad \chi_-^{(1)}\chi_-^{(2)}$$

- calculate the eigenvalues

$$\begin{aligned} S_z\chi_+^{(1)}\chi_+^{(2)} &= \left(S_{1z}+S_{2z}\right)\chi_+^{(1)}\chi_+^{(2)} \\ &= \left(S_{1z}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}\left(S_{2z}\chi_+^{(2)}\right) \\ &= \hbar\chi_+^{(1)}\chi_+^{(2)} \end{aligned}$$

$$S_z\chi_+^{(1)}\chi_-^{(2)} = S_z\chi_-^{(1)}\chi_+^{(2)} = 0 \qquad S_z\chi_-^{(1)}\chi_-^{(2)} = -\hbar\chi_-^{(1)}\chi_-^{(2)}$$

- Two *m*=0 states

spin triplet and singlet

- Spin triplet S=1, m=1, 0, -1
- Spin singlet S=0, m=0
- May check using lowering operator $S_- = S_{1-} + S_{2-}$

$$S_{1-}\chi_+^{(1)} = \hbar\chi_-^{(1)}$$
$$S_{2-}\chi_+^{(2)} = \hbar\chi_-^{(2)}$$

$$S_-\chi_+^{(1)}\chi_+^{(2)} = \left(S_{1-}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}\left(S_{2-}\chi_+^{(2)}\right)$$
$$= \hbar\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$

- S=1, m=0 state $X_+ = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$

spin triplet and singlet

- One may check the result again

$$S_- \frac{\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}}{\sqrt{2}} = \left(S_{1-} + S_{2-}\right)\frac{\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(S_{1-}\chi_+^{(1)}\right)\chi_-^{(2)} + \frac{1}{\sqrt{2}}\chi_-^{(1)}\left(S_{2-}\chi_+^{(2)}\right)$$

$$= \sqrt{2}\hbar\chi_-^{(1)}\chi_-^{(2)}$$

- The remaining state m=0

$$X_- = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} - \chi_+^{(1)}\chi_-^{(2)}\right)$$

S^2

- check the S^2 value

$$\mathbf{S}^2 = \left(\mathbf{S}_1 + \mathbf{S}_2\right)^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$$
$$= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z}$$
$$= \mathbf{S}_1^2 + \mathbf{S}_2^2 + S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}$$

$$\mathbf{S}_1^2 X_+ = \frac{1}{\sqrt{2}}\mathbf{S}_1^2\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$
$$= \frac{3}{4}\hbar^2\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right) = \frac{3}{4}\hbar^2 X_+$$

$$\mathbf{S}_2^2 X_+ = \frac{3}{4}\hbar^2 X_+$$

$$\mathbf{S}_1^2 X_- = \frac{3}{4}\hbar^2 X_-$$

$$\mathbf{S}_2^2 X_- = \frac{3}{4}\hbar^2 X_-$$

$$S_{1z}S_{2z}X_+ = \frac{1}{\sqrt{2}}S_{1z}S_{2z}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$
$$= \frac{1}{\sqrt{2}}S_{1z}\chi_-^{(1)}S_{2z}\chi_+^{(2)} + \frac{1}{\sqrt{2}}S_{1z}\chi_+^{(1)}S_{2z}\chi_-^{(2)}$$
$$= -\frac{1}{4}\hbar^2\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right) = -\frac{1}{4}\hbar^2 X_+$$

$$S_{1z}S_{2z}X_- = -\frac{1}{4}\hbar^2 X_-$$

S^2

$$\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)X_{+}=\frac{1}{\sqrt{2}}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)\left(\chi_{-}^{(1)}\chi_{+}^{(2)}+\chi_{+}^{(1)}\chi_{-}^{(2)}\right)$$

$$=\frac{1}{\sqrt{2}}\left(S_{1+}\chi_{-}^{(1)}\right)\left(S_{2-}\chi_{+}^{(2)}\right)+\frac{1}{\sqrt{2}}\left(S_{1-}\chi_{+}^{(1)}\right)\left(S_{2+}\chi_{-}^{(2)}\right)$$

$$=\frac{1}{\sqrt{2}}\hbar^{2}\left(\chi_{+}^{(1)}\chi_{-}^{(2)}+\chi_{-}^{(1)}\chi_{+}^{(2)}\right)=\hbar^{2}X_{+}$$

$$\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)X_{-}=\frac{1}{\sqrt{2}}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)\left(\chi_{-}^{(1)}\chi_{+}^{(2)}-\chi_{+}^{(1)}\chi_{-}^{(2)}\right)$$

$$=\frac{1}{\sqrt{2}}\left(S_{1+}\chi_{-}^{(1)}\right)\left(S_{2-}\chi_{+}^{(2)}\right)-\frac{1}{\sqrt{2}}\left(S_{1-}\chi_{+}^{(1)}\right)\left(S_{2+}\chi_{-}^{(2)}\right)$$

$$=-\frac{1}{\sqrt{2}}\hbar^{2}\left(\chi_{+}^{(1)}\chi_{-}^{(2)}-\chi_{-}^{(1)}\chi_{+}^{(2)}\right)=-\hbar^{2}X_{-}$$

S^2

- For X_+ , S=1

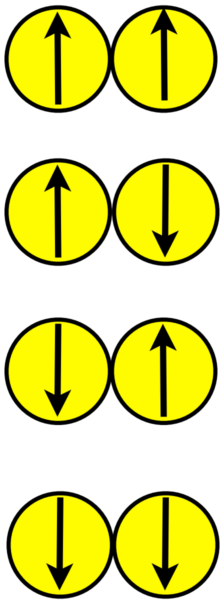
$$\mathbf{S}^2 X_+ = \mathbf{S}_1^2 X_+ + \mathbf{S}_2^2 X_+ + S_{1+} S_{2-} X_+ + S_{1-} S_{2+} X_+ + 2 S_{1z} S_{2z} X_+$$
$$= \frac{3}{4}\hbar^2 X_+ + \frac{3}{4}\hbar^2 X_+ + \hbar^2 X_+ - \frac{1}{2}\hbar^2 X_+$$
$$= 2\hbar^2 X_+ = S(S+1)\hbar^2 X_+$$

- For X_- , S=0

$$\mathbf{S}^2 X_- = \mathbf{S}_1^2 X_- + \mathbf{S}_2^2 X_- + S_{1+} S_{2-} X_- + S_{1-} S_{2+} X_- + 2 S_{1z} S_{2z} X_-$$
$$= \frac{3}{4}\hbar^2 X_- + \frac{3}{4}\hbar^2 X_- - \hbar^2 X_- - \frac{1}{2}\hbar^2 X_-$$
$$= 0$$

representation

- product states



$\chi_+^{(1)}\chi_+^{(2)}$

$\chi_+^{(1)}\chi_-^{(2)}$

$\chi_-^{(1)}\chi_+^{(2)}$

$\chi_-^{(1)}\chi_-^{(2)}$

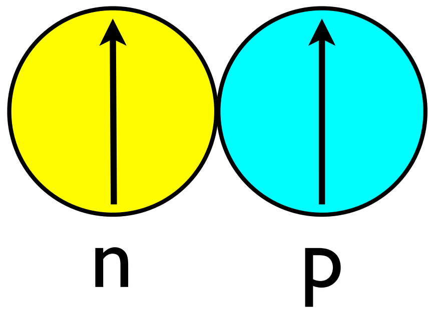
- total spin state

Spin triplet	Spin singlet
S=1	S=0
$\chi_+^{(1)}\chi_+^{(2)}$	
$\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)}+\chi_+^{(1)}\chi_-^{(2)}\right)$	$\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)}-\chi_+^{(1)}\chi_-^{(2)}\right)$
$\chi_-^{(1)}\chi_-^{(2)}$	

spin-dependent potential

- In many physical systems, two particle interaction is spin-dependent
- the duetron hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V_1(r) + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 V_2(r)$$



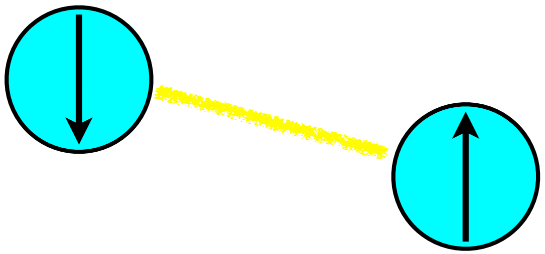
$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right) = \frac{1}{2}\mathbf{S}^2 - \frac{3}{4}\hbar^2$$

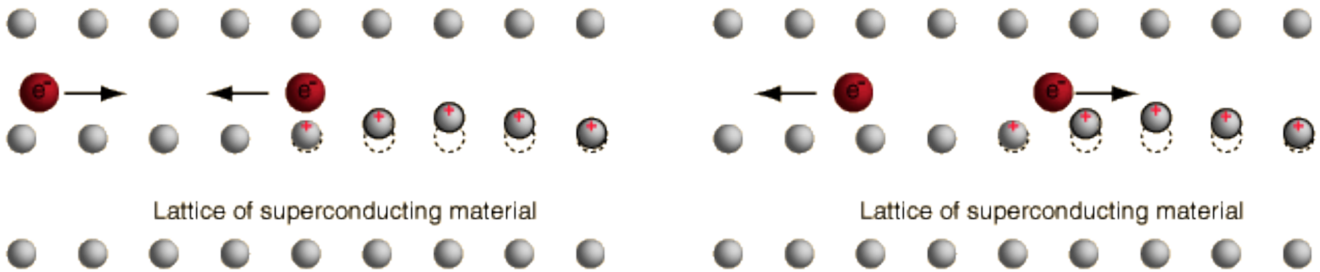
- S^2 is a good quantum number, but S_z is not
- for triplet $V(r) = V_1(r) + \left(1 - \frac{3}{4}\right)V_2(r) = V_1(r) + \frac{1}{4}V_2(r)$
- for singlet $V(r) = V_1(r) + \left(0 - \frac{3}{4}\right)V_2(r) = V_1(r) - \frac{3}{4}V_2(r)$

spin-dependent potential

- for deutron, one observes a bound S=1 state and an unbound S=0 state

- for BCS paring, bound state S=0

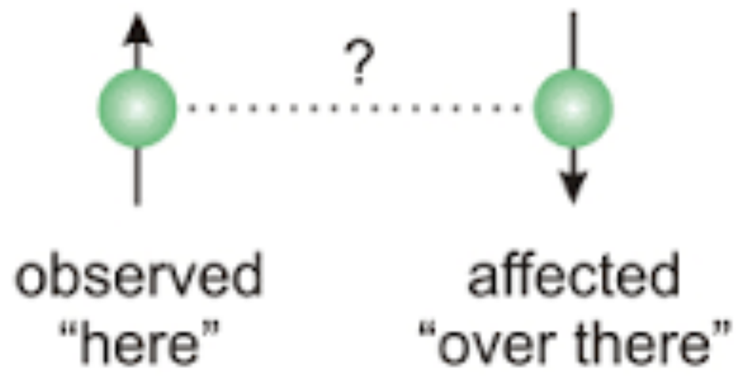




http://hyperphysics.phy-astr.gsu.edu/hbase/Solids/coop.html

spin singlet and entanglement

- In the spin singlet, quantum states are entangled
- First we do S_x measurement on electron 1, we have 50% to get `+’ and 50% to get `-’
- then we do S_x measurement on electron 2, the result is 100% opposite to the result of electron 1.



How does it work?

- entangled state $\psi = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right)$

- the measurement of S_{x1} project the state to an eigenstate of S_{x1}

$$\left| S_x = + \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_{x1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The project operator

$$\begin{aligned} P_1(+) &= \left| S_x = + \right\rangle \left\langle S_x = + \right| \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

measurement

- Projection result

$$P_1(+)\psi = \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2$$

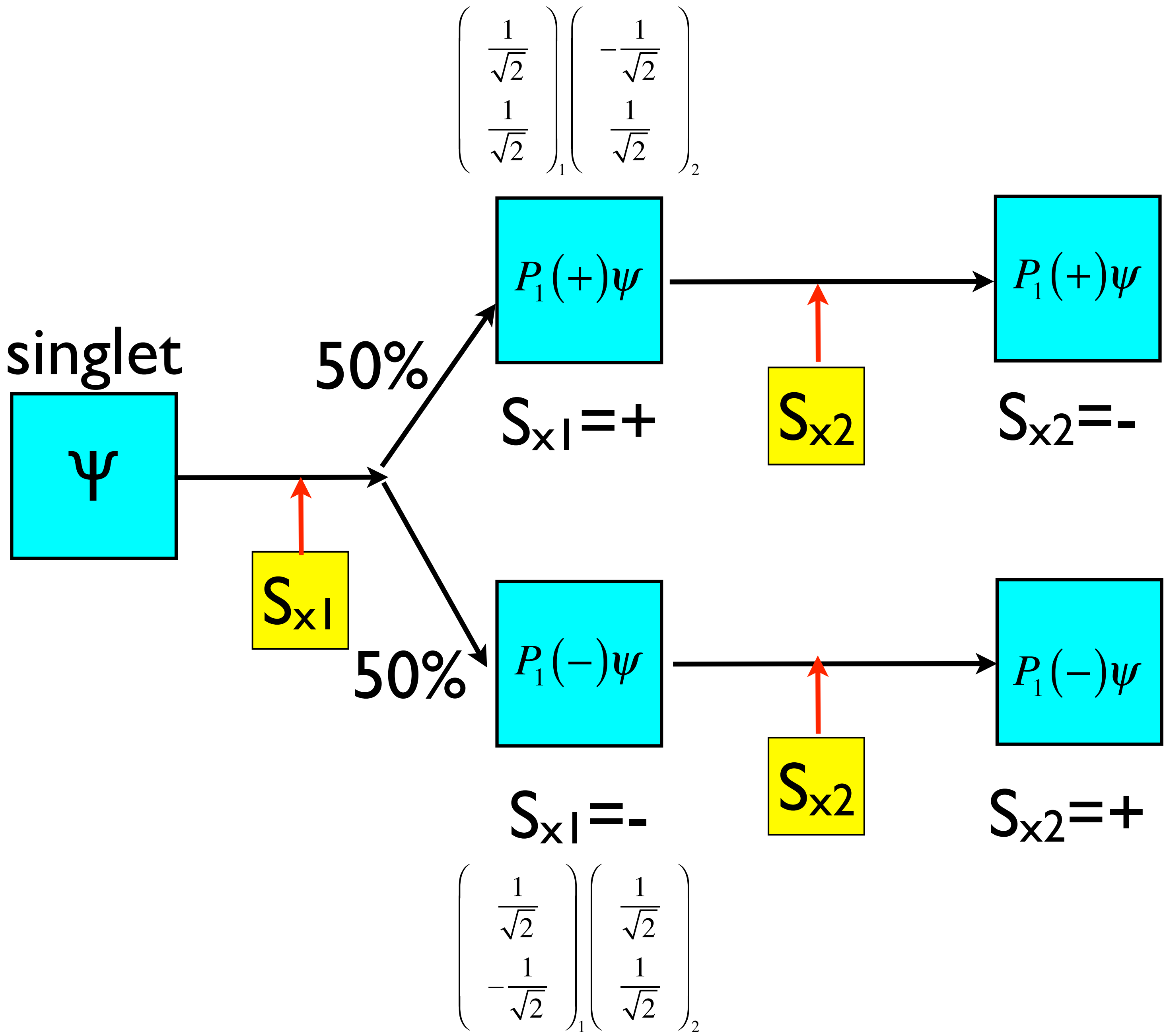
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}_2$$

$$= \psi'$$

- The following measurement on S_{x2} will only give `-' result

$$S_{x2}\psi' = S_{x2}P_1(+)\psi = -\frac{\hbar}{2}\psi'$$



singlet
Ψ
Sx1
50%
50%
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_1 \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_2$
$P_1(+)\psi$
Sx1=+
Sx2
$P_1(+)\psi$
Sx2=-
$P_1(-)\psi$
Sx1=-
Sx2
$P_1(-)\psi$
Sx2=+
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_2$

- Einstein’s comment: “spukhafte Fernwirkung” or "spooky action at a distance

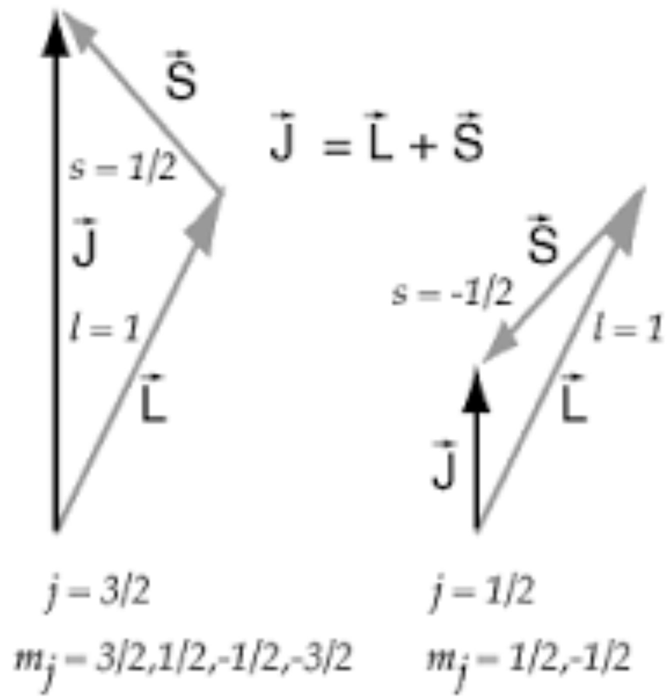
Addition of L and S

- total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$
- product state $Y_{lm}\chi_{\pm}$
- eigenstate $\mathbf{J}^2\psi_{j,m_j} = \hbar^2 j(j+1)\psi_{j,m_j}$

$$J_z\psi_{j,m_j} = \hbar m_j \psi_{j,m_j}$$

- eigenvalue $j = l \pm \frac{1}{2}$

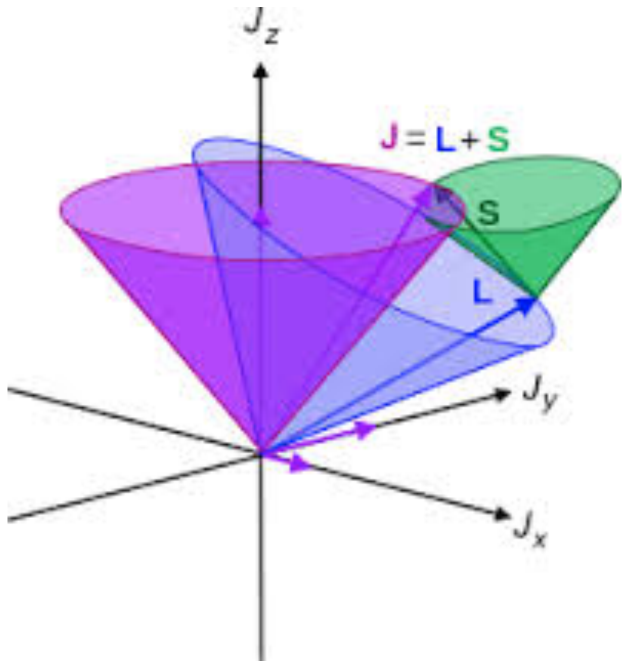
$$m_j = -j, -j+1 \cdots, j-1, j$$



Addition of L and S

- case 1 $j = l + \frac{1}{2}$ $m_j = m + \frac{1}{2}$

$$\psi_{j,m_j} = \sqrt{\frac{l+m+1}{2l+1}} Y_{lm}\chi_+ + \sqrt{\frac{l-m}{2l+1}} Y_{lm+1}\chi_-$$



- case 2 $j = l - \frac{1}{2}$ $m_j = m + \frac{1}{2}$

$$\psi_{j,m_j} = \sqrt{\frac{l-m}{2l+1}} Y_{lm}\chi_+ + \sqrt{\frac{l+m+1}{2l+1}} Y_{lm+1}\chi_-$$

Addition of angular momenta

$$\mathbf{J} = \mathbf{L}_1 + \mathbf{L}_2$$

- possible total angular momentum

$$j = l_1 + l_2, l_1 + l_2 - 1, \cdots |l_1 - l_2|$$

- possible z-component

$$m_j = -j, -j + 1 \cdots, j - 1, j$$