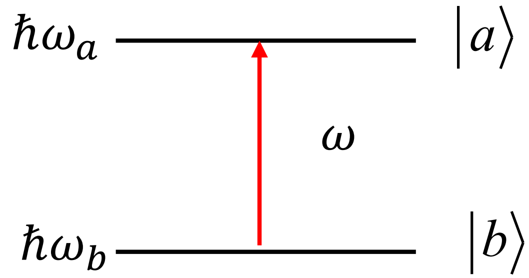
Quantum optics

Two level system

- An two-level atom in an external time-varying electric field



- Atom Hamiltonian $H_0 = \begin{pmatrix} \hbar\omega_a & 0 \\ 0 & \hbar\omega_b \end{pmatrix}$

- Perturbation: E-field in x direction $H_1 = eE_0 x \cos \omega t$

- Matrix element $\langle a | H_1 | a \rangle = eE_0 \langle a | x | a \rangle \cos \omega t$

- The parity symmetry $\langle a | x | a \rangle = \langle b | x | b \rangle = 0$

Interaction picture

- The perturbation can be written as

$$H_1 = \begin{pmatrix} 0 & W \cos\omega t \\ W^* \cos\omega t & 0 \end{pmatrix} \qquad W = eE_0 \langle a | x | b \rangle$$

- The interaction picture states that

$$|\psi_I(t)\rangle = e^{\frac{i}{\hbar}H_0 t} |\psi(t)\rangle$$

- Schrodinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}(H_0 + H_1)|\psi(t)\rangle$$

Interaction picture

- Schrodinger equation

$$\frac{d}{dt}\left|\psi_I(t)\right\rangle = \frac{d}{dt}\left(e^{\frac{i}{\hbar}H_0 t}\left|\psi(t)\right\rangle\right)$$

$$= \frac{i}{\hbar}H_0 e^{\frac{i}{\hbar}H_0 t}\left|\psi(t)\right\rangle + e^{\frac{i}{\hbar}H_0 t}\frac{d}{dt}\left|\psi(t)\right\rangle$$

$$= \frac{i}{\hbar}H_0 e^{\frac{i}{\hbar}H_0 t}\left|\psi(t)\right\rangle - \frac{i}{\hbar}e^{\frac{i}{\hbar}H_0 t}\left(H_0 + H_1\right)\left|\psi(t)\right\rangle$$

$$= -\frac{i}{\hbar}e^{\frac{i}{\hbar}H_0 t}H_1\left|\psi(t)\right\rangle = -\frac{i}{\hbar}e^{\frac{i}{\hbar}H_0 t}H_1 e^{-\frac{i}{\hbar}H_0 t}e^{\frac{i}{\hbar}H_0 t}\left|\psi(t)\right\rangle$$

$$= -\frac{i}{\hbar}V(t)e^{\frac{i}{\hbar}H_0 t}\left|\psi_I(t)\right\rangle$$

$$\frac{d}{dt}\left|\psi_I(t)\right\rangle = -\frac{i}{\hbar}V(t)\left|\psi_I(t)\right\rangle \qquad V(t) = e^{\frac{i}{\hbar}H_0 t}H_1 e^{-\frac{i}{\hbar}H_0 t}$$

Time dependence

- To calculate

$$e^{\frac{i}{\hbar}H_0 t} = \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \qquad \left(H_0\right)^n = \begin{pmatrix} \left(\hbar\omega_a\right)^n & 0 \\ 0 & \left(\hbar\omega_b\right)^n \end{pmatrix}$$

$$\begin{aligned} V(t) &= \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \begin{pmatrix} 0 & W\cos\omega t \\ W^*\cos\omega t & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_a t} & 0 \\ 0 & e^{-i\omega_b t} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \begin{pmatrix} 0 & We^{-i\omega_b t}\cos\omega t \\ W^* e^{-i\omega_a t}\cos\omega t & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & We^{i(\omega_a-\omega_b)t}\cos\omega t \\ W^* e^{-i(\omega_a-\omega_b)t}\cos\omega t & 0 \end{pmatrix} \end{aligned}$$

$$e^{i(\omega_a-\omega_b)t}\cos\omega t = \frac{1}{2}e^{i(\omega_a-\omega_b-\omega)t} + \frac{1}{2}e^{i(\omega_a-\omega_b+\omega)t}$$

Close to resonance

- When the driving frequency is close to the level spacing

$$\omega_a - \omega_b - \omega = \Delta \ll \omega$$

$$e^{i(\omega_a - \omega_b)t} \cos \omega t = \frac{1}{2} e^{i(\omega_a - \omega_b - \omega)t} + \frac{1}{2} e^{i(\omega_a - \omega_b + \omega)t}$$

slow oscillating   fast oscillating

- Rotational wave approximation: the fast oscillating term can be dropped

$$V(t) = \begin{pmatrix} 0 & \frac{W}{2} e^{i\Delta t} \\ \frac{W^*}{2} e^{-i\Delta t} & 0 \end{pmatrix}$$

Differential equation

- Parameterize the wavefunction

$$|\psi_I(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

- The equations

$$\frac{da(t)}{dt} = -\frac{i}{\hbar}\frac{W}{2}e^{i\Delta t}b(t)$$

$$\frac{db(t)}{dt} = -\frac{i}{\hbar}\frac{W^*}{2}e^{-i\Delta t}a(t)$$

- Single equation

$$\frac{d^2a(t)}{dt^2} - i\Delta\frac{da(t)}{dt} + \frac{|W|^2}{4\hbar^2}a(t) = 0$$

Rabi frequency

- The solution is assumed $a(t) = e^{i\Omega t}$

$$\Omega^2 - \Delta\Omega - \frac{|W|^2}{4\hbar^2} = 0$$

$$\Omega = \frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + \frac{|W|^2}{4\hbar^2}}$$

- The general solution

$$a(t) = a_1 e^{i\Omega_1 t} + a_2 e^{i\Omega_2 t}$$

$$b(t) = \frac{2\hbar}{|W|}\left(\Omega_1 a_1 e^{-i\Omega_2 t} + \Omega_2 a_2 e^{-i\Omega_1 t}\right)$$

Initial state

- If the atom is initially at the upper state

$$a(0) = a_1 + a_2 = 1$$

$$b(0) = \frac{2\hbar}{|W|}\left(\Omega_1 a_1 + \Omega_2 a_2\right) = 0$$

$$a_1 = \frac{\Omega_2}{\Omega_2 - \Omega_1} \qquad a_2 = \frac{\Omega_1}{\Omega_1 - \Omega_2}$$

- The probability amplitude in the lower state

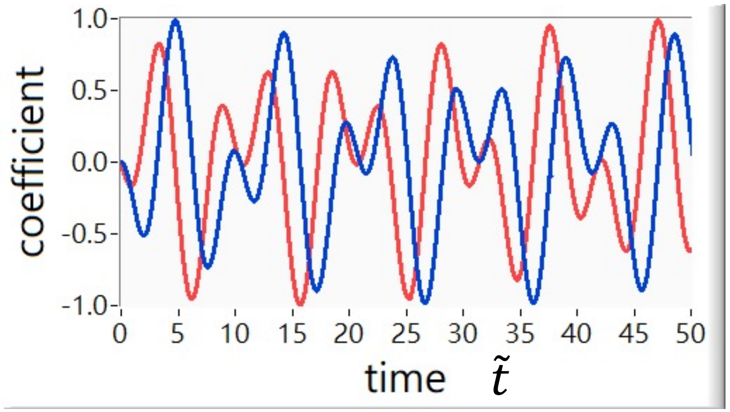
$$b(t) = \frac{2\hbar}{|W|}\frac{\Omega_1\Omega_2}{\Omega_1 - \Omega_2}\left(e^{-i\Omega_2 t} - e^{-i\Omega_1 t}\right)$$

$$\hbar\omega_{ba} = \hbar\omega_b - \hbar\omega_a$$ to be energy unit

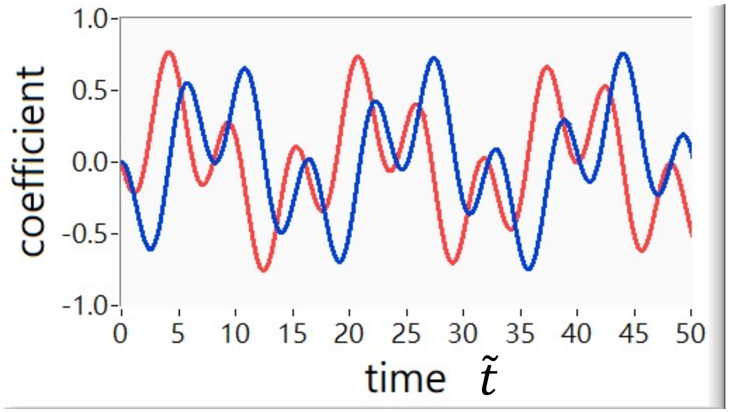
Dimensionless parameters $\tilde{\Delta} = \Delta/\omega_{ba}$ $\tilde{t} = \omega_{ba} t$

$$\tilde{W} = \frac{W}{\hbar\omega_{ba}}$$

$\tilde{\Delta} = 0, \tilde{W} = 0.6$



$\tilde{\Delta} = 0.5, \tilde{W} = 0.6$



Probability

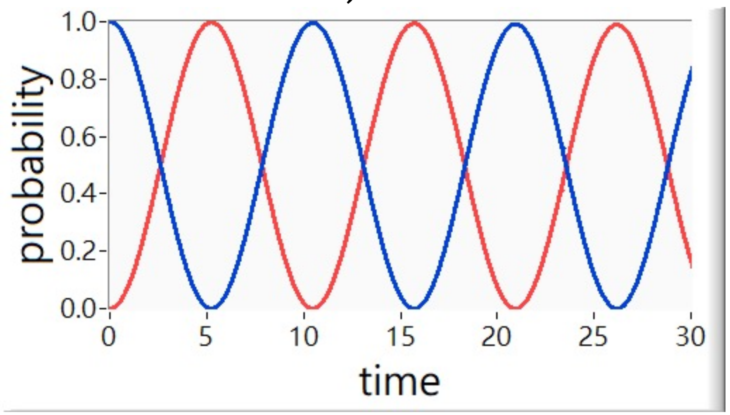
- Calculate the probability

$$P(t)=\left|b(t)\right|^2=\frac{4\hbar^2}{|W|^2}\left(\frac{\Omega_1\Omega_2}{\Omega_1-\Omega_2}\right)^2\left|e^{-i\Omega_2 t}-e^{-i\Omega_1 t}\right|^2$$

$$=\frac{4\hbar^2}{|W|^2}\left(\frac{\Omega_1\Omega_2}{\Omega_1-\Omega_2}\right)^2\sin^2\frac{(\Omega_1-\Omega_2)}{2}t$$

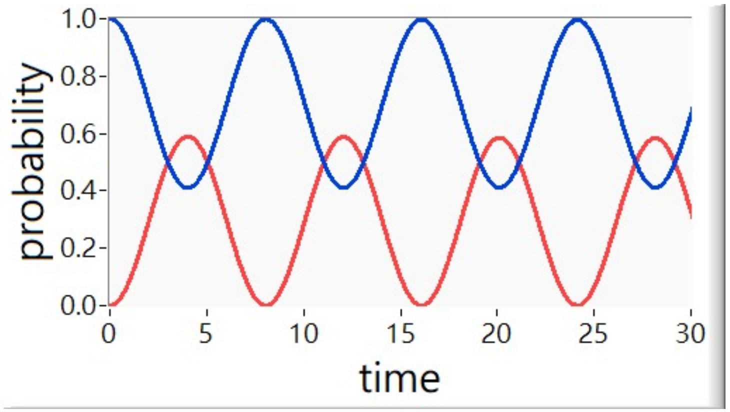
$$=\frac{|W|^2}{|W|^2+\hbar^2\Delta^2}\sin^2\sqrt{\frac{\Delta^2}{4}+\frac{|W|^2}{4\hbar^2}}t$$

$\tilde{\Delta}=0, \tilde{W}=0.6$



$\tilde{t}=\omega_{ba}t$

$\tilde{\Delta}=0.5, \tilde{W}=0.6$



Quantum treatment of E-field

- To write E-field as an operator
- Basis: the direct product of atom and photon state

$$H_0 = \begin{pmatrix} \hbar\omega_a + N\hbar\omega & 0 \\ 0 & \hbar\omega_b + (N+1)\hbar\omega \end{pmatrix} \qquad \begin{matrix} |a\rangle \otimes |N\rangle \\ |b\rangle \otimes |N+1\rangle \end{matrix}$$

- Interaction

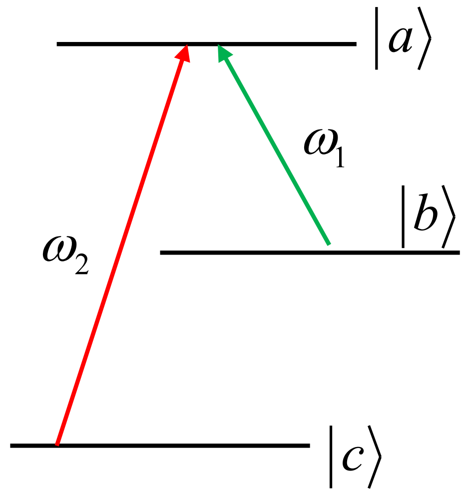
$$W = e\langle N|E_0|N+1\rangle\langle a|x|b\rangle = \sqrt{N+1}W_0 \qquad E_0 \to \left(a + a^+\right)$$

- Rabi model

$$H = \hbar\omega_a\sigma_z + \hbar\omega a^+ a + g\left(a + a^+\right)\sigma_x$$

3 level system

- Atom Hamiltonian $H_0 = \begin{pmatrix} \hbar\omega_a & 0 & 0 \\ 0 & \hbar\omega_b & 0 \\ 0 & 0 & \hbar\omega_c \end{pmatrix}$



- Two-tone: $H_1 = eE_1 x\cos\omega_1 t + eE_2 x\cos\omega_2 t$

- Driving perturbation

$$H_1 = \begin{pmatrix} 0 & W_{1ab}\cos\omega_1 t + W_{2ab}\cos\omega_2 t & W_{1ac}\cos\omega_1 t + W_{2ac}\cos\omega_2 t \\ W_{1ab}{}^{*}\cos\omega_1 t + W_{2ab}{}^{*}\cos\omega_2 t & 0 & 0 \\ W_{1ac}{}^{*}\cos\omega_1 t + W_{2ac}{}^{*}\cos\omega_2 t & 0 & 0 \end{pmatrix}$$

Interaction picture and RWA

- In the interaction picture

$$V = \begin{pmatrix} 0 & X & Y \\ X^* & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix}$$

$$X = \left(W_{1ab}\cos\omega_1 t + W_{2ab}\cos\omega_2 t\right)e^{i(\omega_a-\omega_b)t}$$

$$Y = \left(W_{1ac}\cos\omega_1 t + W_{2ac}\cos\omega_2 t\right)e^{i(\omega_a-\omega_c)t}$$

- Detuning

$$\delta_1 = \omega_a - \omega_b - \omega_1$$

$$\delta_2 = \omega_a - \omega_c - \omega_2$$

$$X = \frac{W_{1ab}}{2}e^{i\delta_1 t}$$

$$Y = \frac{W_{2ac}}{2}e^{i\delta_2 t}$$

solution

- state

$$\left|\psi_I(t)\right\rangle = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$$

$$\frac{da(t)}{dt} = -\frac{i}{\hbar}\frac{W_{1ab}}{2}e^{i\delta_1 t}b(t) - \frac{i}{\hbar}\frac{W_{2ac}}{2}e^{i\delta_2 t}c(t)$$

$$\frac{db(t)}{dt} = -\frac{i}{\hbar}\frac{W_{1ab}{}^{*}}{2}e^{-i\delta_1 t}a(t)$$

$$\frac{dc(t)}{dt} = -\frac{i}{\hbar}\frac{W_{2ac}{}^{*}}{2}e^{-i\delta_2 t}a(t)$$

$$a(t) = Ae^{i\Omega t}$$

$$b(t) = Be^{i\Omega t}e^{-i\delta_1 t}$$

$$c(t) = Ce^{i\Omega t}e^{-i\delta_2 t}$$

$$-\hbar\begin{pmatrix} \Omega & 0 & 0 \\ 0 & \Omega-\delta_1 & 0 \\ 0 & 0 & \Omega-\delta_2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}{}^{*}}{2} & 0 & 0 \\ \frac{W_{2ac}{}^{*}}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Eigen modes

$$\begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^{\ *}}{2} & -\hbar\delta_1 & 0 \\ \frac{W_{2ac}^{\ *}}{2} & 0 & -\hbar\delta_2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = -\hbar\Omega \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Omega(\Omega - \delta_1)(\Omega - \delta_2) - \frac{|W_{2ac}|^2}{4\hbar^2}(\Omega - \delta_1) - \frac{|W_{1ab}|^2}{4\hbar^2}(\Omega - \delta_2) = 0$$

Perfect tuning

- In resonance $\delta_1 = \delta_2 = 0$

$$\Omega^3 - \left(\frac{|W_{1ab}|^2}{4\hbar^2} + \frac{|W_{2ac}|^2}{4\hbar^2}\right)\Omega = 0$$

- Three modes

$$\Omega = 0, \pm\frac{1}{2\hbar}\sqrt{|W_{1ab}|^2 + |W_{2ac}|^2}$$

Eigen modes for perfect tuning

- zero frequency mode

$$\left|\psi_I(t)\right\rangle = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \frac{1}{\sqrt{\left|W_{1ab}\right|^2 + \left|W_{2ac}\right|^2}} \begin{pmatrix} 0 \\ W_{2ac} \\ -W_{1ab} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & 0 & 0 \\ \frac{W_{2ac}^*}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

- Non-zero mode

$$\left|\psi_I(t)\right\rangle = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\Omega t} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{W_{1ab}^*}{2\hbar\Omega} \\ -\frac{W_{2ac}^*}{2\hbar\Omega} \end{pmatrix} e^{i\Omega t}$$

$$\begin{pmatrix} \Omega & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & \Omega & 0 \\ \frac{W_{2ac}^*}{2} & 0 & \Omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

Superposition of quantum states

- General solution

$$\left|\psi_I(t)\right\rangle = \frac{\alpha}{2\hbar\Omega_0}\begin{pmatrix} 0 \\ W_{2ac} \\ -W_{1ab} \end{pmatrix} + \frac{\beta}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{W_{1ab}{}^*}{2\hbar\Omega_0} \\ -\frac{W_{2ac}{}^*}{2\hbar\Omega_0} \end{pmatrix} e^{i\Omega_0 t} + \frac{\gamma}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{W_{1ab}{}^*}{2\hbar\Omega_0} \\ \frac{W_{2ac}{}^*}{2\hbar\Omega_0} \end{pmatrix} e^{-i\Omega_0 t}$$

- Initial state

$$\left|\psi_I(0)\right\rangle = \begin{pmatrix} \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \\ \frac{W_{2ac}}{2\hbar\Omega_0}\alpha + \frac{W_{1ab}{}^*}{2\hbar\Omega_0}\left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}}\right) \\ \frac{-W_{1ab}}{2\hbar\Omega_0}\alpha + \frac{W_{2ac}{}^*}{2\hbar\Omega_0}\left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}}\right) \end{pmatrix}$$

Dark state

- When the initial state is

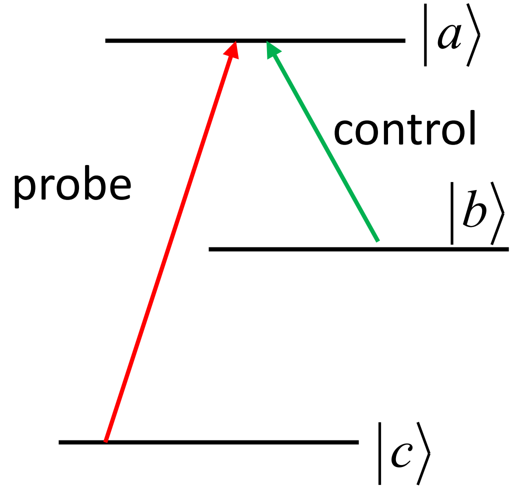
$$\left|\psi_I(0)\right\rangle = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \qquad \tan\theta = \frac{-W_{2ac}}{W_{1ab}}$$

- The state is fully at the zero frequency mode, and the state is constant of time

$$\left|\psi_I(t)\right\rangle = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$$

- The state is not excited by the EM waves, and is called dark state

Electromagnetically induced transparency



- If $W_{2ac} \ll W_{1ab}$
- The initial state at $|\psi_I(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\beta + \gamma = 0$$

$$W_{2ac}\alpha + W_{1ab}{}^* \left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \right) = W_{2ac}\alpha - \sqrt{2} W_{1ab}{}^* \beta = 0$$

$$\frac{-W_{1ab}\alpha - \sqrt{2} W_{2ac}{}^* \beta}{2\hbar\Omega_0} = 1 \qquad \alpha = \frac{W_{1ab}{}^*}{2\hbar\Omega_0} \qquad \beta = \frac{W_{2ac}}{\sqrt{2} W_{1ab}{}^*}\alpha$$

- Probability at $|\alpha\rangle$

$$\langle \alpha | \psi_I(t) \rangle = \frac{\beta}{\sqrt{2}} \left(e^{i\Omega_0 t} - e^{-i\Omega_0 t} \right) = \frac{1}{\sqrt{2}} \frac{W_{2ac}}{\sqrt{2} W_{1ab}{}^*} \frac{W_{1ab}{}^*}{2\hbar\Omega_0} \left(2i \sin \Omega_0 t \right) = \frac{iW_{2ac}}{2\hbar\Omega_0} \sin \Omega_0 t$$

$$P_a = \frac{|W_{2ac}|^2}{|W_{2ac}|^2 + |W_{1ab}|^2} \sin^2 \Omega_0 t \ll 1$$