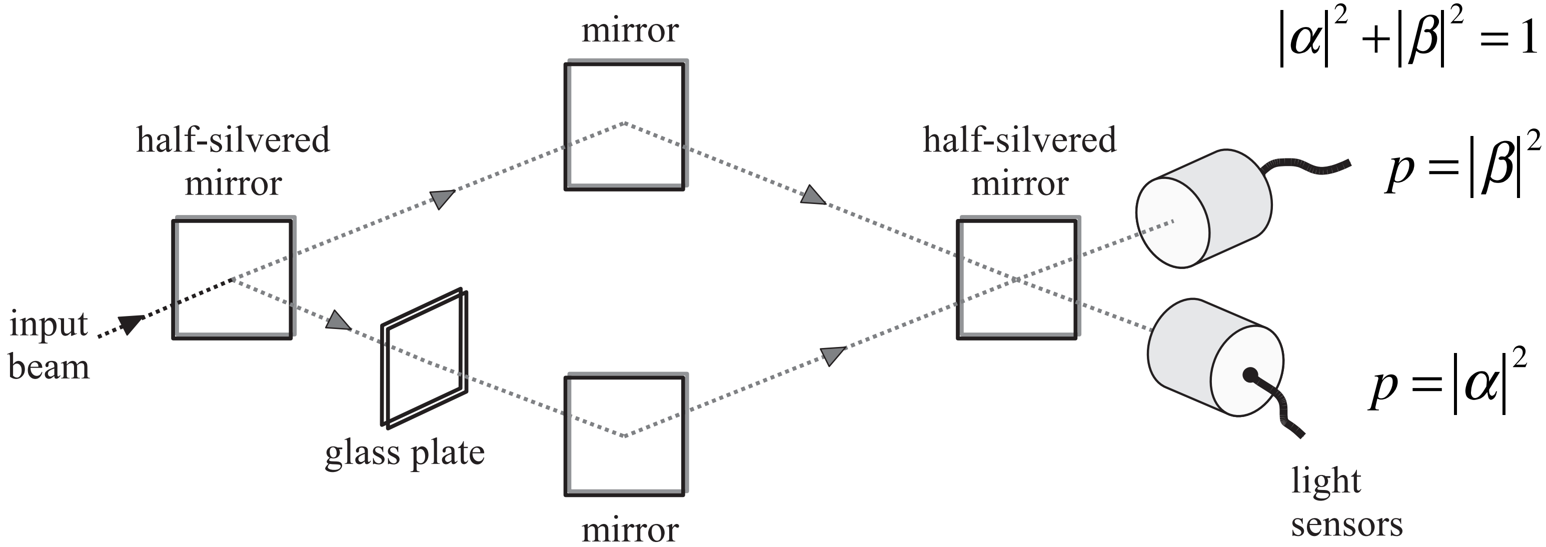
Two level system

Interferometers

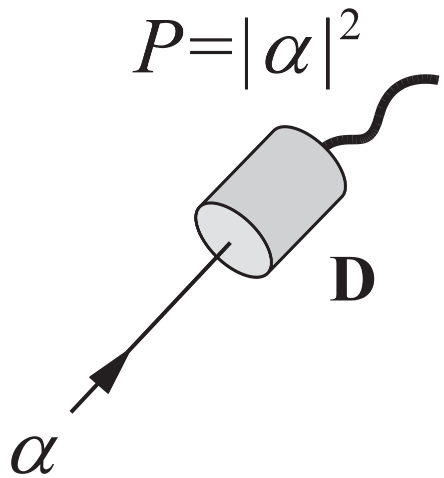
- Mach–Zehnder interferometer

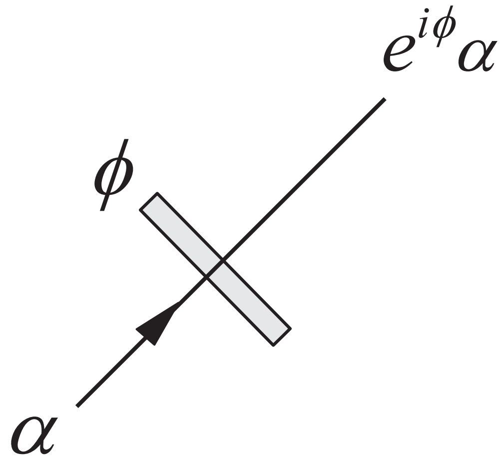


- Light in the input beam is divided into two beams, which are later recombined. Light sensors measure the intensities of the two output beams

phase shifter

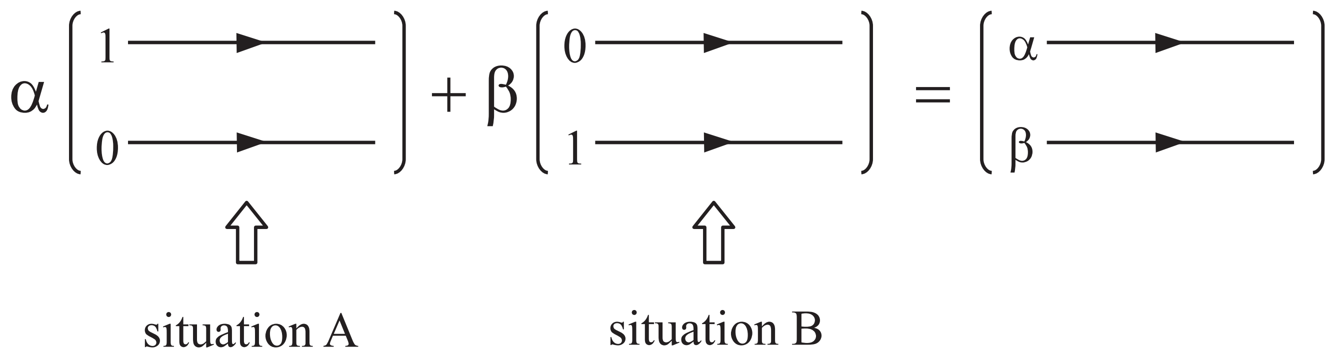
- A phase shifter alter α to $e^{i\varphi}\alpha$ without altering the probability that the photon is found in the beam.





principle of superposition

- In situation A, the photon is certainly in the upper beam



- Given complex coefficients α and β, then there is a possible physical situation which we can formally write as

$$\alpha\ (\text{situation } A) + \beta\ (\text{situation } B)\ .$$

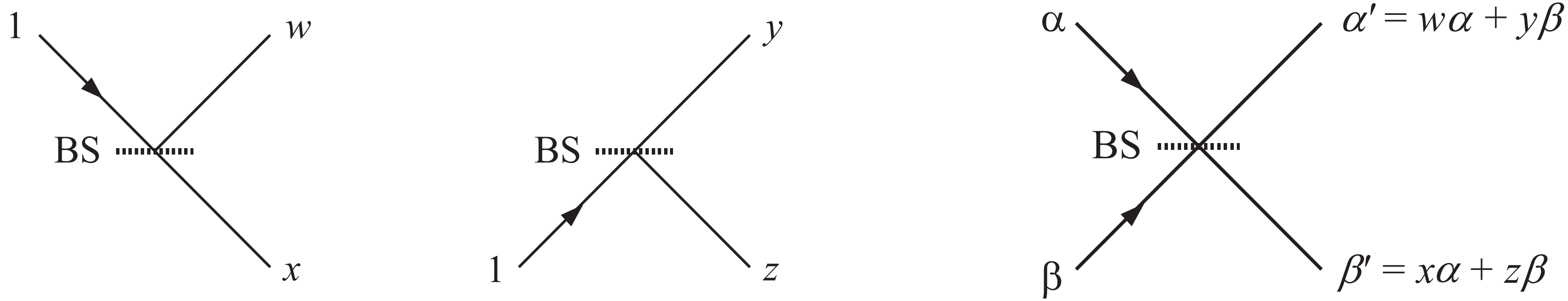
- describe each situation by a column vector

- situation A $\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ situation B $\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- superposed state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

beamsplitter

- input beams of unit amplitude produce output beams with amplitudes w, x, y, and z



- express in the amplitude-vector notation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} w & y \\ x & z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Probability conservation

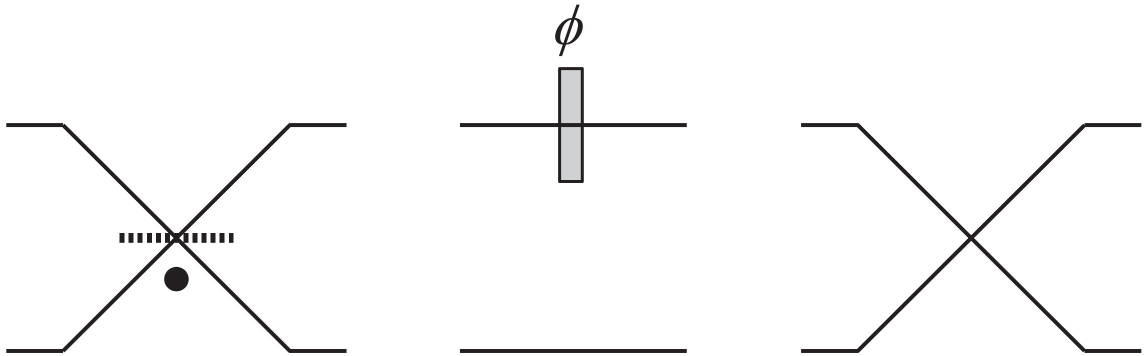
- the probability that the photon is reflected or transmitted are the same

$$|w|^2 = |x|^2 = |y|^2 = |z|^2 = \frac{1}{2}$$

- conservation of probability requires that if constructive interference happens in some places, destructive interference must happen elsewhere

$$|\alpha'|^2 + |\beta'|^2 = 1 \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad R^\dagger R = 1$$

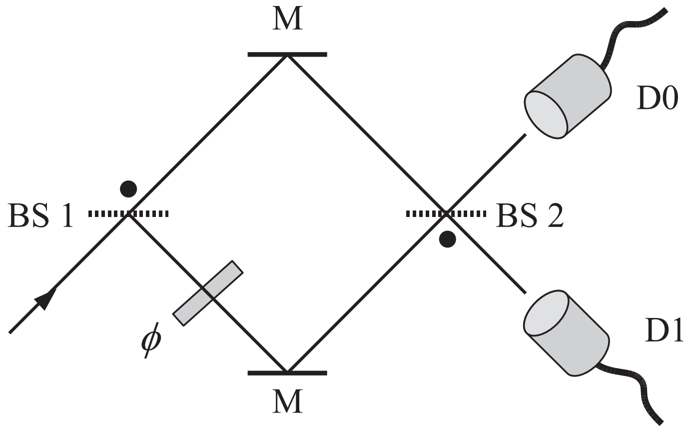
matrix representation

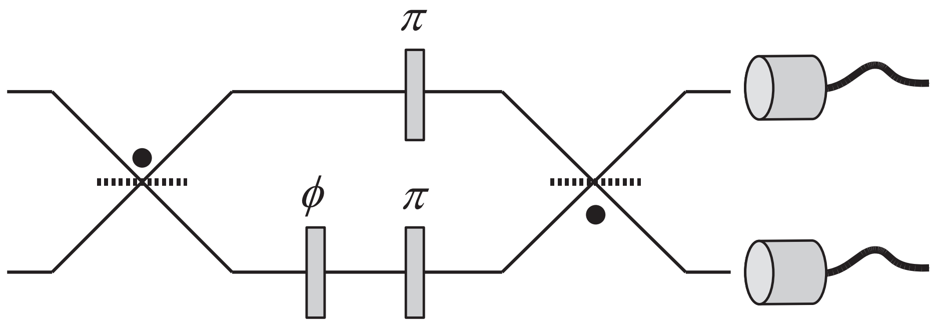


$\mathbf{B}_l$ $\mathbf{P}_u(\phi)$ $\mathbf{X}$

$$B_l = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad P_u = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

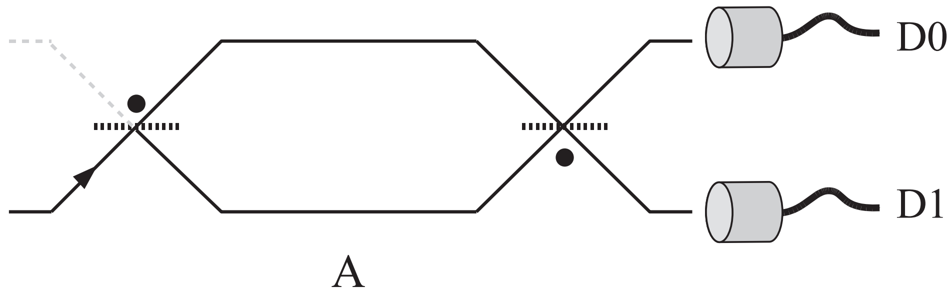
Mach–Zehnder interferometer





interference revisit

- consider the simplified Mach–Zehnder arrangement



- the matrix representation of the apparatus

$$B_l B_u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

input amplitude $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

outcome	*P*	
photon reaches D0	1	constructive
photon reaches D1	0.	destructive

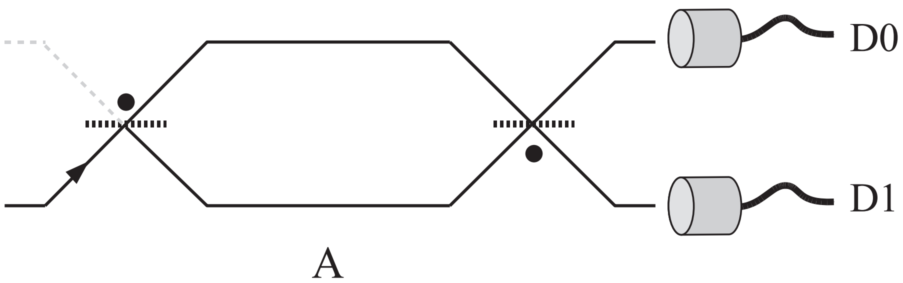
- if we send a photon into the apparatus, it has a 50% probability of striking an absorber A

outcome	*P*
photon reaches D0	1/4
photon reaches D1	1/4
photon hits hand	1/2.

- by blocking one beam with A, we have actually increased the probability that the photon is detected by D1

Elitzur–Vaidman thought-experiment

- If the bomb explodes, then it was in working order, but this bomb is now lost. If the photon is detected by D0, the test is inconclusive and may be repeated. But if the photon ever arrives at D1, then the managers know that the unexploded bomb is in working order, even though the bomb never detects the passage of the photon.



Bomb is a dud

outcome	*P*
photon reaches D0	1
photon reaches D1	0
bomb explodes	0

Bomb is working

outcome	*P*
photon reaches D0	1/4
photon reaches D1	1/4
bomb explodes	1/2.

spin 1/2 system

- A particle may have an intrinsic angular momentum called spin
- Electrons, protons, and neutrons are all examples of spin-1/2 particles
- If one measure the z-component S_z(or S_x, S_y) of the spin angular momentum for one of these particles, he gets

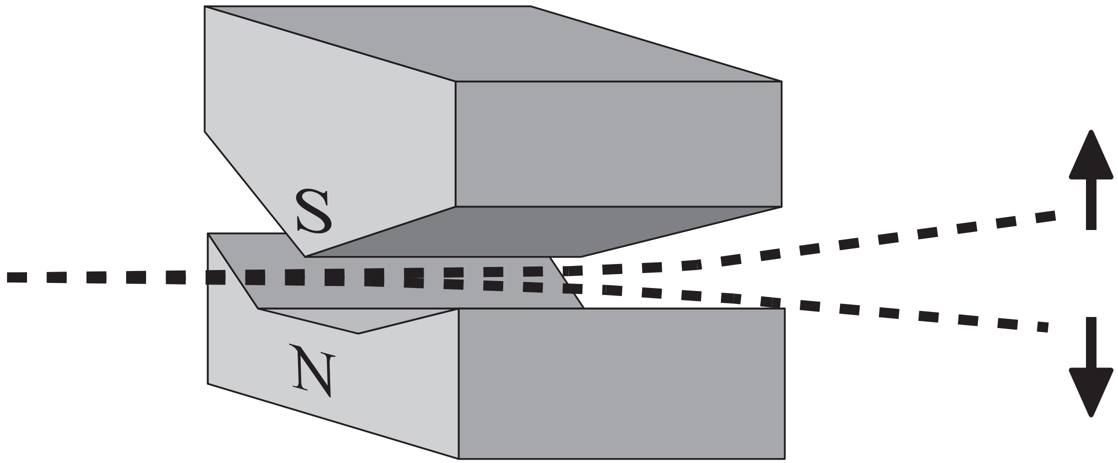
$$S_z = \pm \frac{\hbar}{2}$$

Stern-Gerlach experiment

- A stream of atoms moving from the right passes between the asymmetric poles of a magnet. Particles with different values of μ_z are deflected in different directions. The final position of the atom determines its μ_z

$\vec{\mu} = \gamma \vec{S}$ γ is gyromagnetic ratio

$E = -\vec{\mu} \cdot \vec{B}$



the spin state

- superpositions of spin-up and spin-down states

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha |z_+\rangle + \beta |z_-\rangle$$

Bloch sphere

$$|x_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{1}{\sqrt{2}}|z_-\rangle$$

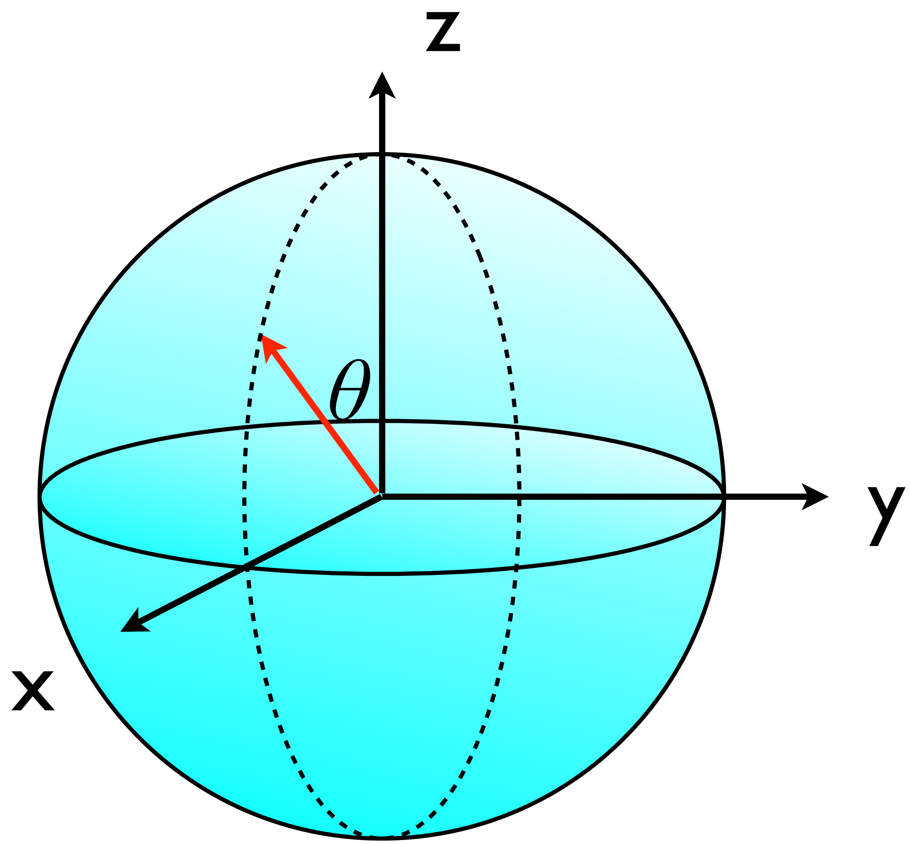
$$|x_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle$$

why? $\langle x_-|x_+\rangle = 0$

$$|\langle z_+|x_+\rangle|^2 = |\langle z_-|x_+\rangle|^2 = \frac{1}{2}$$

$$|y_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{i}{\sqrt{2}}|z_-\rangle$$

$$|y_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{i}{\sqrt{2}}|z_-\rangle$$



Pauli matrices

- Hermitian operators in 2 level systems $\mathbf{S} = \frac{1}{2}\hbar\sigma$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Commutation relations

$$\left[\sigma_x, \sigma_y\right] = 2i\sigma_z \qquad \left[S_x, S_y\right] = i\hbar S_z$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

- They are anti-commute

$$\left\{\sigma_a, \sigma_b\right\} = 2\delta_{ab}$$

eigenstates of S_x

- To find the eigenstates for $S_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- The eigenequation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$

- The eigenevalue $\lambda^2 - 1 = 0$ $\lambda = \pm 1$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Projection operator

- the projection to +x and -x direction

$$|x_+\rangle\langle x_+| = \left(\frac{1}{\sqrt{2}}|z_+\rangle + \frac{1}{\sqrt{2}}|z_-\rangle\right)\left(\frac{1}{\sqrt{2}}\langle z_+| + \frac{1}{\sqrt{2}}\langle z_-|\right)$$
$$= \frac{1}{2}\left(|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-| + |z_+\rangle\langle z_-| + |z_-\rangle\langle z_+|\right)$$

$$|x_+\rangle\langle x_+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|x_-\rangle\langle x_-| = \left(\frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle\right)\left(\frac{1}{\sqrt{2}}\langle z_+| - \frac{1}{\sqrt{2}}\langle z_-|\right)$$
$$= \frac{1}{2}\left(|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-| - |z_+\rangle\langle z_-| - |z_-\rangle\langle z_+|\right)$$

$$|x_-\rangle\langle x_-| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

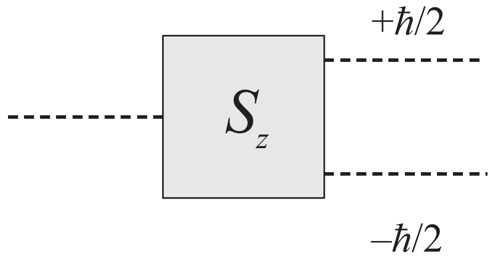
$$P_{x\pm}{}^2 = P_{x\pm}$$

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\left(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|\right) = \frac{\hbar}{2}\left(|z_+\rangle\langle z_-| + |z_-\rangle\langle z_+|\right)$$

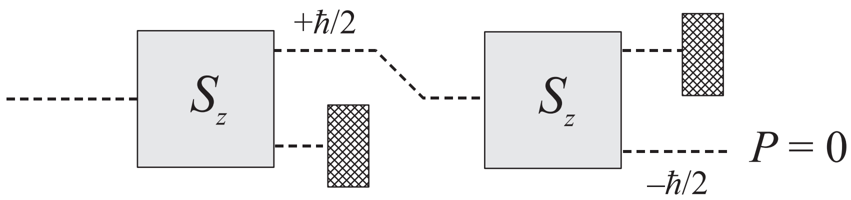
$$\langle S_x\rangle = \frac{\hbar}{2}\langle\psi|\sigma_x|\psi\rangle$$

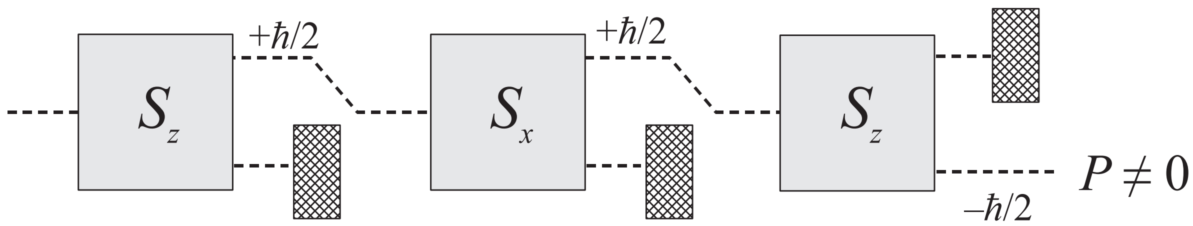
spin filters

Stern–Gerlach apparatus



Stern–Gerlach filters





- S_z and S_x are complementary quantities

Bloch sphere

eigenstates of Sz

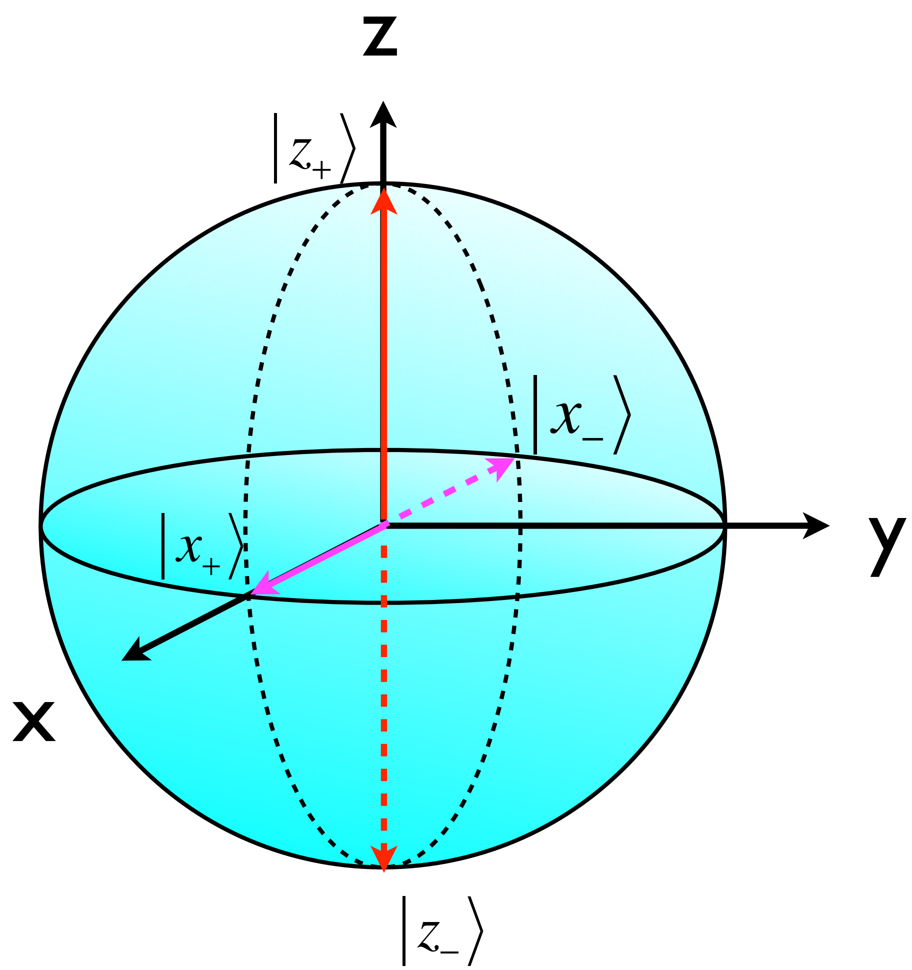
$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

eigenstates of Sx

$$|x_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{1}{\sqrt{2}}|z_-\rangle$$

$$|x_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle$$

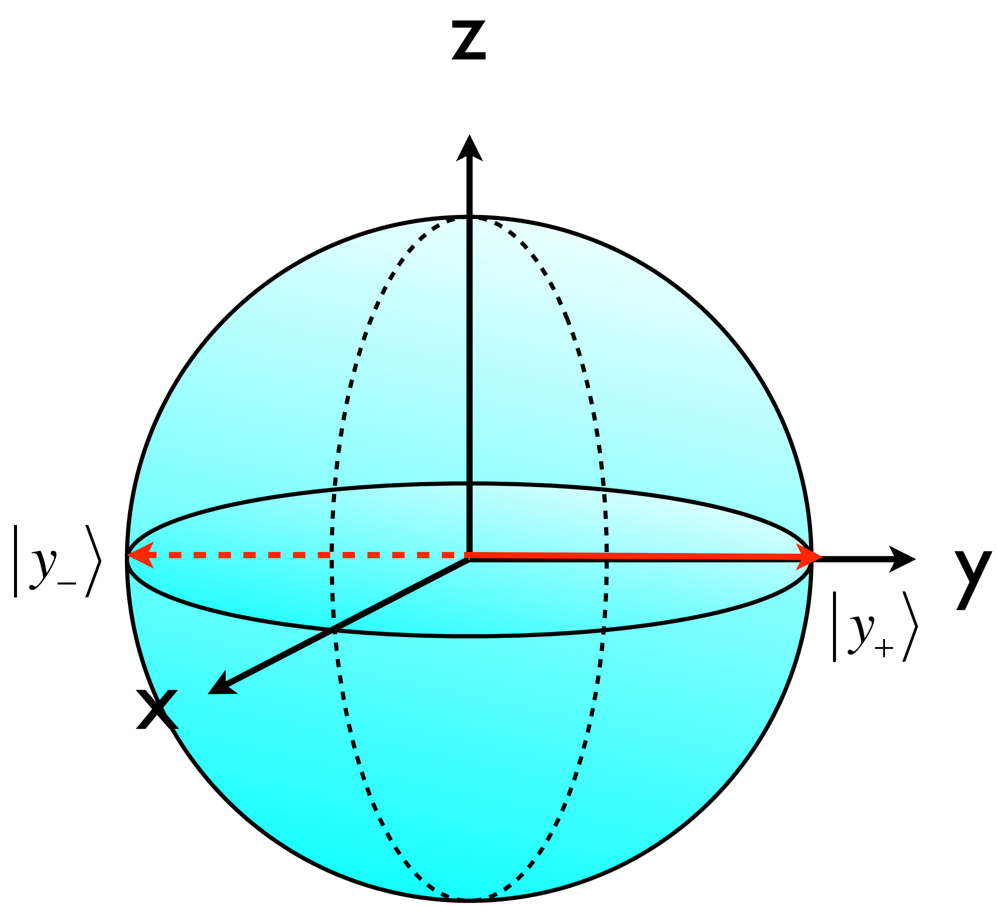


Bloch sphere

eigenstates of Sy

$$|y_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{i}{\sqrt{2}}|z_-\rangle$$

$$|y_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{i}{\sqrt{2}}|z_-\rangle$$



some eigenstates

- To find the eigenstates for

$$S_\theta = S_z \cos\theta + S_x \sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

- The eigenequation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

- The eigenevalue $\lambda^2 - 1 = 0$ $\quad \lambda = \pm 1$
- for $\lambda = 1$ $\quad \cos\theta u + \sin\theta v = u$

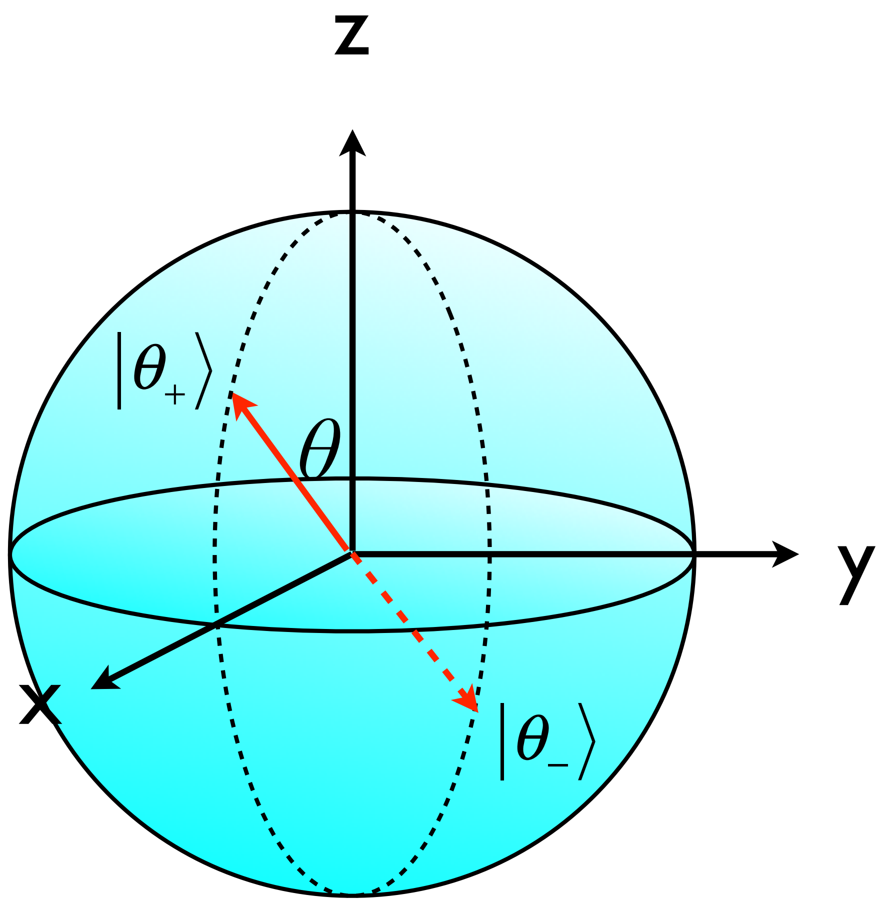
$$|\theta_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}|z_+\rangle + \sin\frac{\theta}{2}|z_-\rangle \qquad |\theta_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} = \sin\frac{\theta}{2}|z_+\rangle - \cos\frac{\theta}{2}|z_-\rangle$$

rotation in θ

- Suppose we choose a direction in the xz-plane that is inclined at an angle θ from the z-axis. Then the amplitude vectors

$$|\theta_+\rangle = \cos\frac{\theta}{2}|z_+\rangle + \sin\frac{\theta}{2}|z_-\rangle$$

$$|\theta_-\rangle = \sin\frac{\theta}{2}|z_+\rangle - \cos\frac{\theta}{2}|z_-\rangle$$



more eigenstates

- To find the eigenstates for

$$S_\phi = S_x \cos\phi + S_y \sin\phi = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

- The eigenequation

$$\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

- The eigenevalue $\lambda^2 - 1 = 0$ $\lambda = \pm 1$
- for $\lambda = 1$ $u = e^{-i\phi} v$ $\lambda = -1$ $u = -e^{-i\phi} v$

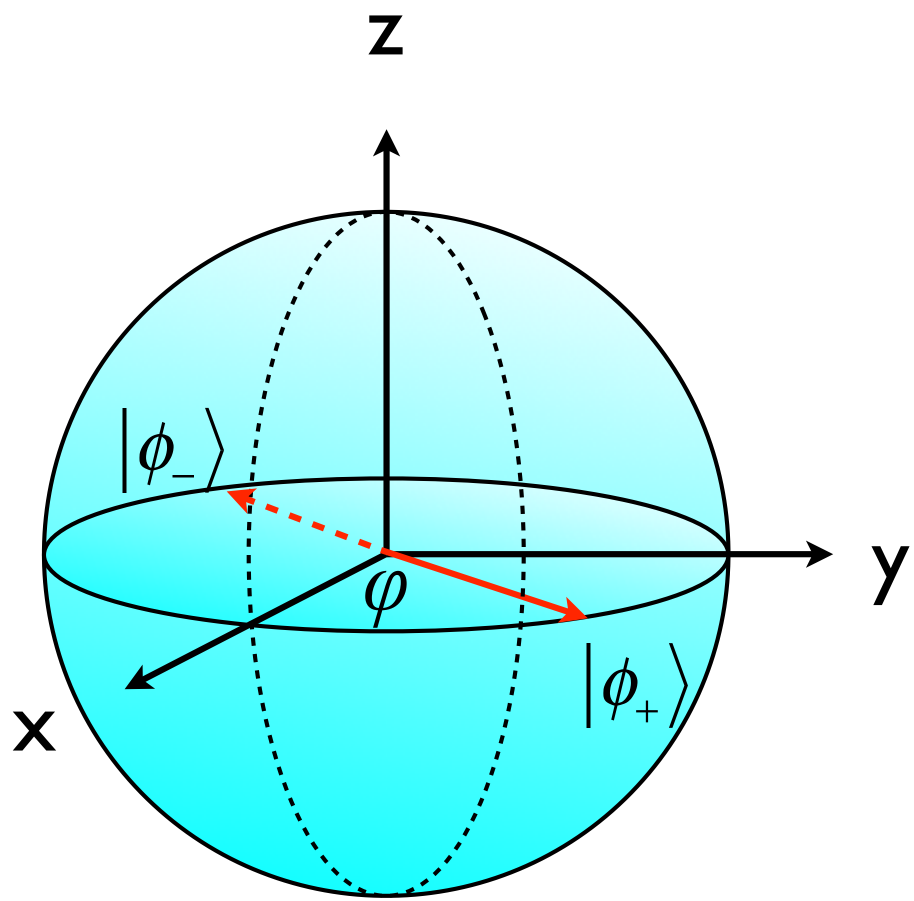
$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \qquad |\phi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{-i\phi} \end{pmatrix}$$

rotation in φ

eigenstates of S_ϕ

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$$

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix}$$

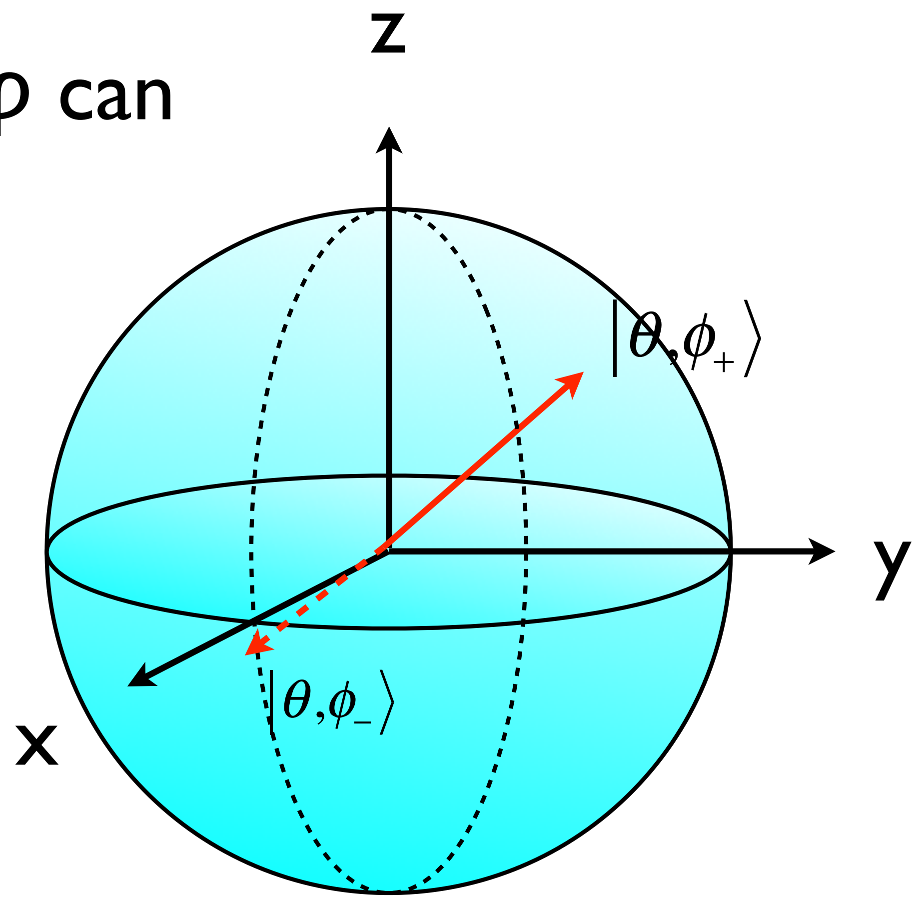


General case

- Any rotation in θ and φ can be shown that

$$|\theta,\phi_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$|\theta,\phi_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{-i\phi} \end{pmatrix}$$



are eigenstates of

$$S_{\theta,\phi} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_x \cos\theta = \mathbf{n}_{\theta,\phi}\cdot\mathbf{S}$$

rotation about z

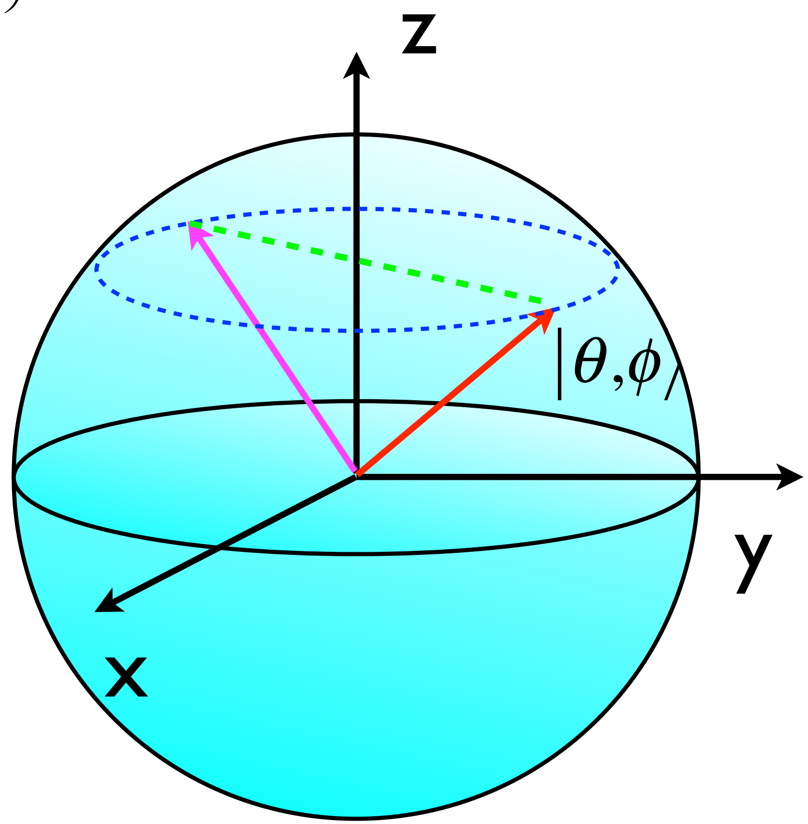
$$\sigma_z|\theta,\phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

$$\theta \to \theta$$

$$\phi \to \pi + \phi$$

can be viewed as the rotation about z of π

also called Pauli-Z gate



rotation about x

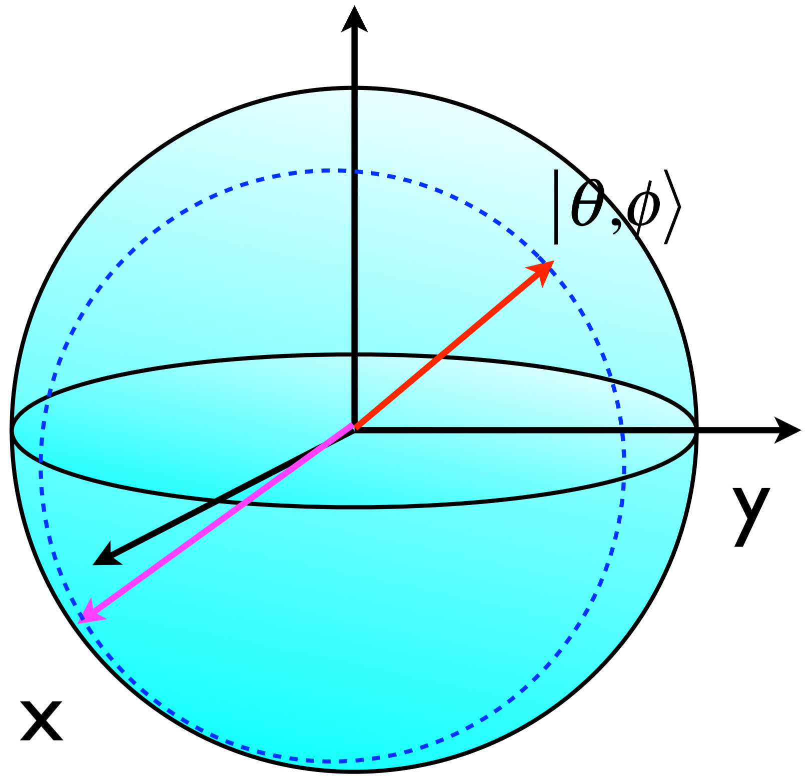
$$\sigma_x|\theta,\phi_+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = e^{i\phi} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{-i\phi} \end{pmatrix}$$

$$\theta \to \pi - \theta$$

$$\phi \to -\phi$$

rotation about x of π

also called Pauli-X gate
or NOT gate



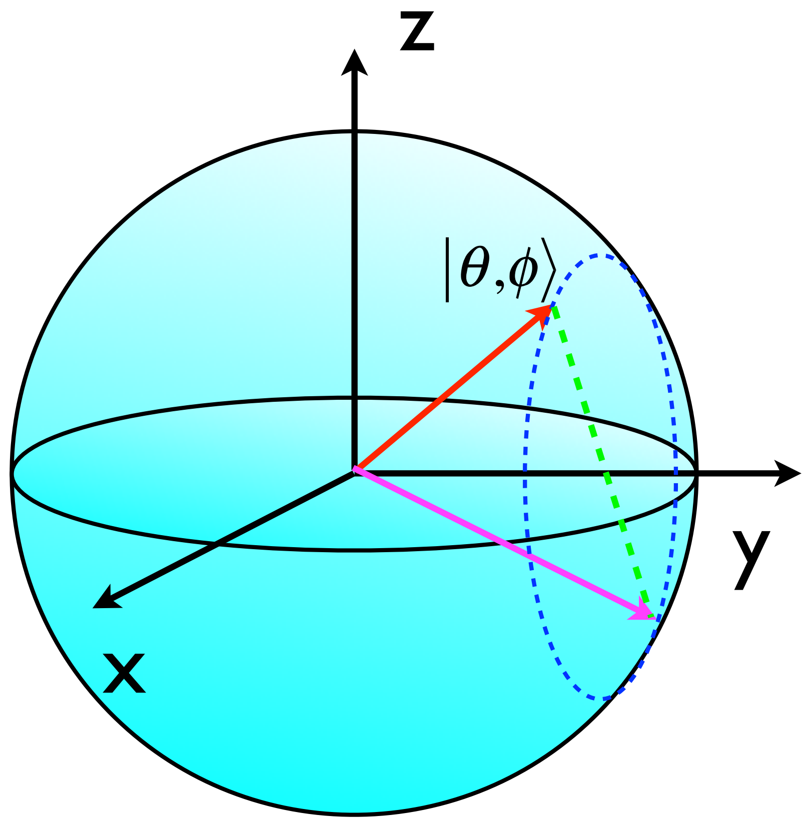
rotate about y

$$\sigma_y|\theta,\phi_+\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \begin{pmatrix} -i\sin\frac{\theta}{2}e^{i\phi} \\ i\cos\frac{\theta}{2} \end{pmatrix} = -ie^{i\phi} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{-i\phi} \end{pmatrix}$$

$$\theta \to \pi - \theta$$
$$\phi \to \pi - \phi$$

rotation about y of π

also called Pauli-Y gate



Gate	Transformation on Bloch sphere (defined for single qubit)
X	π-rotation around the X axis, Z→−Z. Also referred to as a bit-flip.
Z	π-rotation around the Z axis, X→−X. Also referred to as a phase-flip.
H	maps X→Z, and Z→X. This gate is required to make superpositions.
S	maps X→Y. This gate extends H to make complex superpositions. (π/2 rotation around Z axis).
$S^\dagger$	inverse of S. maps X→−Y. (-π/2 rotation around Z axis).
T	π/4 rotation around Z axis.
$T^\dagger$	-π/4 rotation around Z axis.

Hadamard (H) gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- It maps

$$\left|z_+\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longleftrightarrow \left|x_+\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left|z_-\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow \left|x_-\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- for other states, it acts as a rotation about z of π, followed by a rotation about y of π/2

Phase gate

- Phase gates are defined $R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$
- when $\phi = \pi$ $R_\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is Pauli-Z gate
- when $\phi = \frac{\pi}{2}$ $R_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \sqrt{Z}$

  rotation about z of π/2 (called S in IBM Q)
- when $\phi = \frac{\pi}{4}$ $R_{\pi/4} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1+i \end{pmatrix}$

  (called T in IBM Q)

Square root of NOT gate

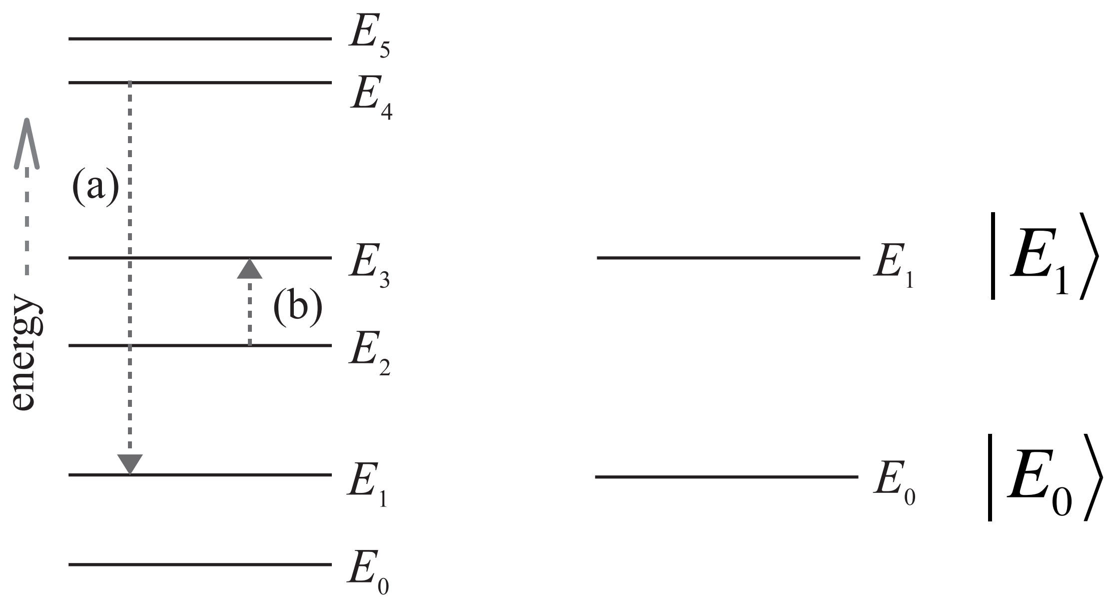
$$\sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

rotation about x of π/2 also called $\sqrt{\text{NOT}}$

Energy levels and quantum states

- An atom generally has many different energy levels. In many experiments only two energy levels – usually the ground state and one excited state – play any significant role. In this case, we can adopt a simplified model, the two-level atom,



Time evolution

- In general, then, the atom will be in a state

$$|\psi\rangle = \alpha|E_0\rangle + \beta|E_1\rangle$$

- at t = 0 the state is $|\psi(0)\rangle = |E_k\rangle$, then at a later time

$$|\psi(t)\rangle = e^{-i\omega_k t}|E_k\rangle \qquad E_k = \hbar\omega_k$$

- probability P_u at time t

$$P_u(t) = |\langle u|\psi(t)\rangle|^2 = |\langle u|\psi(t)\rangle|^2 = P_u$$ stationary states

time evolution

$$|\psi\rangle = \alpha|E_0\rangle + \beta|E_1\rangle \qquad |\psi(t)\rangle = \alpha e^{-i\omega_0 t}|E_0\rangle + \beta e^{-i\omega_1 t}|E_1\rangle$$

- the relative phases of the two terms will change

$$|\psi(0)\rangle = |u\rangle = \frac{1}{\sqrt{2}}|E_0\rangle + \frac{1}{\sqrt{2}}|E_1\rangle \qquad |\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t}|E_0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega_1 t}|E_1\rangle$$

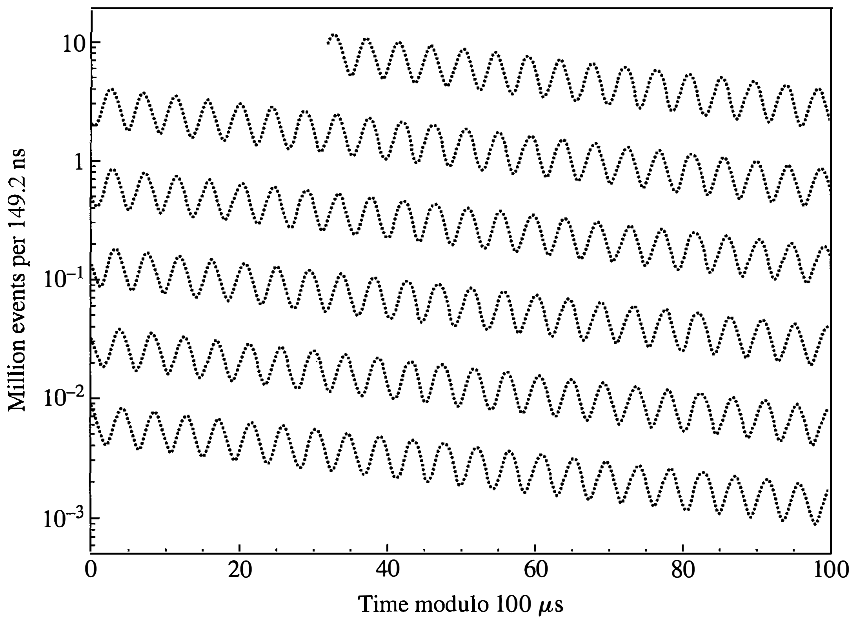
$$\langle u|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega_0 t} + e^{-i\omega_1 t}\right)$$

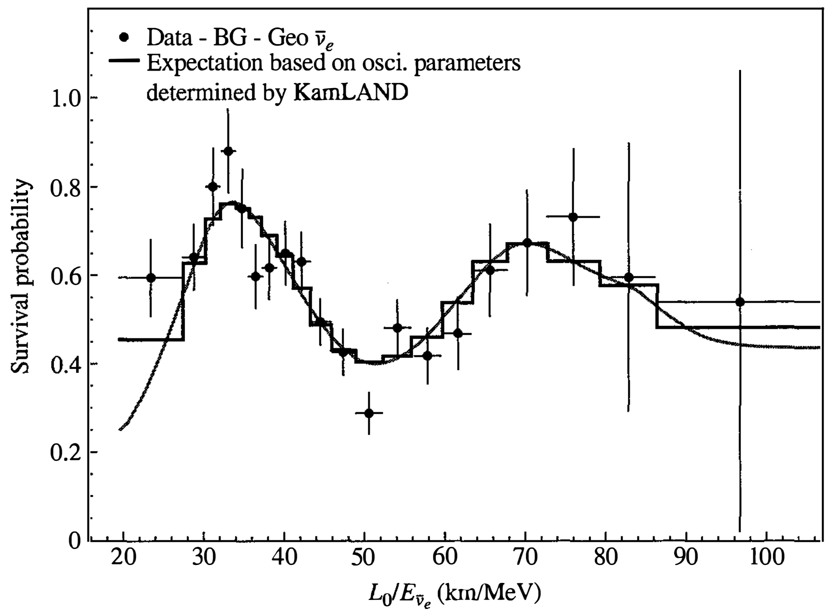
$$A(0) = a(0) = 1$$
$$B(0) = b(0) = 0$$

$$P_u(t) = \left|\langle u|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|e^{-i\omega_0 t} + e^{-i\omega_1 t}\right|^2 = \frac{1}{2}\left(1 + \cos\Delta\omega_0 t\right)$$

- As time progresses, the probability $P_u(t)$ of the measurement outcome u changes from 1 to 0 and then back to 1 again with an angular frequency $\Delta\omega = \omega_1 - \omega_0$

- Precession of muon spin PRD73, 072003(2006)





- Neutrino oscillation PRL100, 221803 (2008)

time evolution operator

- $U(t)\left|E_k\right\rangle = e^{-i\omega_k t}\left|E_k\right\rangle$ for an energy level state $|E_k\rangle$
- U(t) acts on states in a linear way

$$U(t)\left|\psi(0)\right\rangle = \left|\psi(t)\right\rangle$$

- The product of time evolution operators

$$U(t_2) = U(t_2 - t_1)U(t_1)$$

Hamiltonian operator

- $H|E_k\rangle = E_k\,|E_k\rangle$ for an energy level state $|E_k\rangle$
- H acts on states in a linear way.

$$\left|\psi(t)\right\rangle = \alpha e^{-i\omega_0 t}\left|E_0\right\rangle + \beta e^{-i\omega_1 t}\left|E_1\right\rangle$$

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \alpha E_0 e^{-i\omega_0 t}\left|E_0\right\rangle + \beta E_1 e^{-i\omega_1 t}\left|E_1\right\rangle = H\left|\psi(t)\right\rangle$$

- Schrödinger equation

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

spin precession

- If the magnetic field points in the positive z-direction $E = -\gamma B S_z$

- Larmor frequency $\Omega = \gamma B$

$$\left|\psi(0)\right\rangle = \left|x_+\right\rangle = \frac{1}{\sqrt{2}}\left|z_+\right\rangle + \frac{1}{\sqrt{2}}\left|z_-\right\rangle$$

$$\left|\psi(t)\right\rangle = \alpha e^{i\Omega t}\left|z_+\right\rangle + \beta e^{-i\Omega t}\left|z_-\right\rangle$$

$$P_{x+}(t) = \left|\left\langle x_+ \middle| \psi(t)\right\rangle\right|^2 = \frac{1}{2}(1 + \cos\Omega t)$$

nuclear spin resonance

- a proton has a gyromagnetic ratio $\gamma_p = 2.675 \times 10^8 \text{ s}^{-1}\text{T}^{-1}$

- Larmor frequency at B=10T

$$\Omega = \gamma_p B = 2.675 \times 10^9 \text{ s}^{-1}$$

frequency =425.7 MHz

Addition of two spins

- Total spin $\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$
- commutation relation

$$\left[S_x, S_y\right] = \left[S_{1x} + S_{2x}, S_{1y} + S_{2y}\right] = \left[S_{1x}, S_{1y}\right] + \left[S_{2x}, S_{2y}\right]$$

$$= i\hbar S_{1z} + i\hbar S_{2z} = i\hbar S_z$$

- Therefor it is easy to find total spin S satisfies the commutation relation of an angular momentum

product states

- The possible states are (product states)

$$|\uparrow\rangle|\uparrow\rangle \quad |\uparrow\rangle|\downarrow\rangle \quad |\downarrow\rangle|\uparrow\rangle \quad |\downarrow\rangle|\downarrow\rangle$$

- calculate the eigenvalues

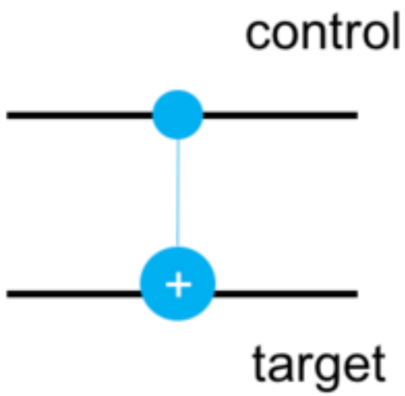
$$\begin{aligned} S_z|\uparrow\rangle|\uparrow\rangle &= \left(S_{1z}+S_{2z}\right)|\uparrow\rangle|\uparrow\rangle \\ &= \left(S_{1z}|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(S_{2z}|\uparrow\rangle\right) \\ &= \hbar|\uparrow\rangle|\uparrow\rangle \end{aligned}$$

$$S_z|\downarrow\rangle|\downarrow\rangle = -\hbar\downarrow\rangle|\downarrow\rangle$$

$$S_z|\uparrow\rangle|\downarrow\rangle = S_z|\downarrow\rangle|\uparrow\rangle = 0$$

- Two zero S_z product states

2-bit gate



Starting state		Ending State
|00>	→	|00>
|10>	→	|10>
|01>	→	|11>
|11>	→	|01>

<u>ibm q: beginer's guide</u>

Entangled states

- product states

$$|\uparrow\rangle|\uparrow\rangle$$

$$|\uparrow\rangle|\downarrow\rangle$$

$$|\downarrow\rangle|\uparrow\rangle$$

$$|\downarrow\rangle|\downarrow\rangle$$

- Bell states

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle\right)$$

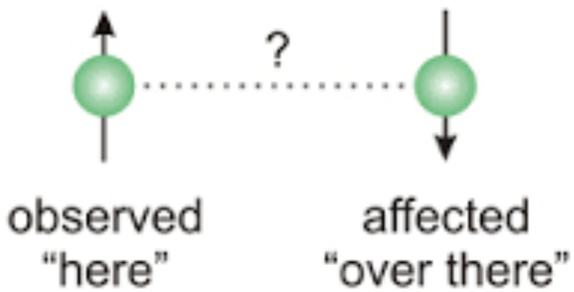
$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

spin entanglement

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right)$$

- First we do S_x measurement on electron 1, we have 50% to get `+' and 50% to get `-'
- then we do S_x measurement on electron 2, the result is 100% same to the result of electron 1.



How does it work?

- entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$

- the measurement of S_{x1} project the state to an eigenstate of S_{x1}

$$S_{1x} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}_1$$

- The project operator $P_{1x}(+) = |x+\rangle\langle x+| = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}_1$

measurement

- Projection result

$$P_{1x}(+)|\Psi\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}_1 \left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

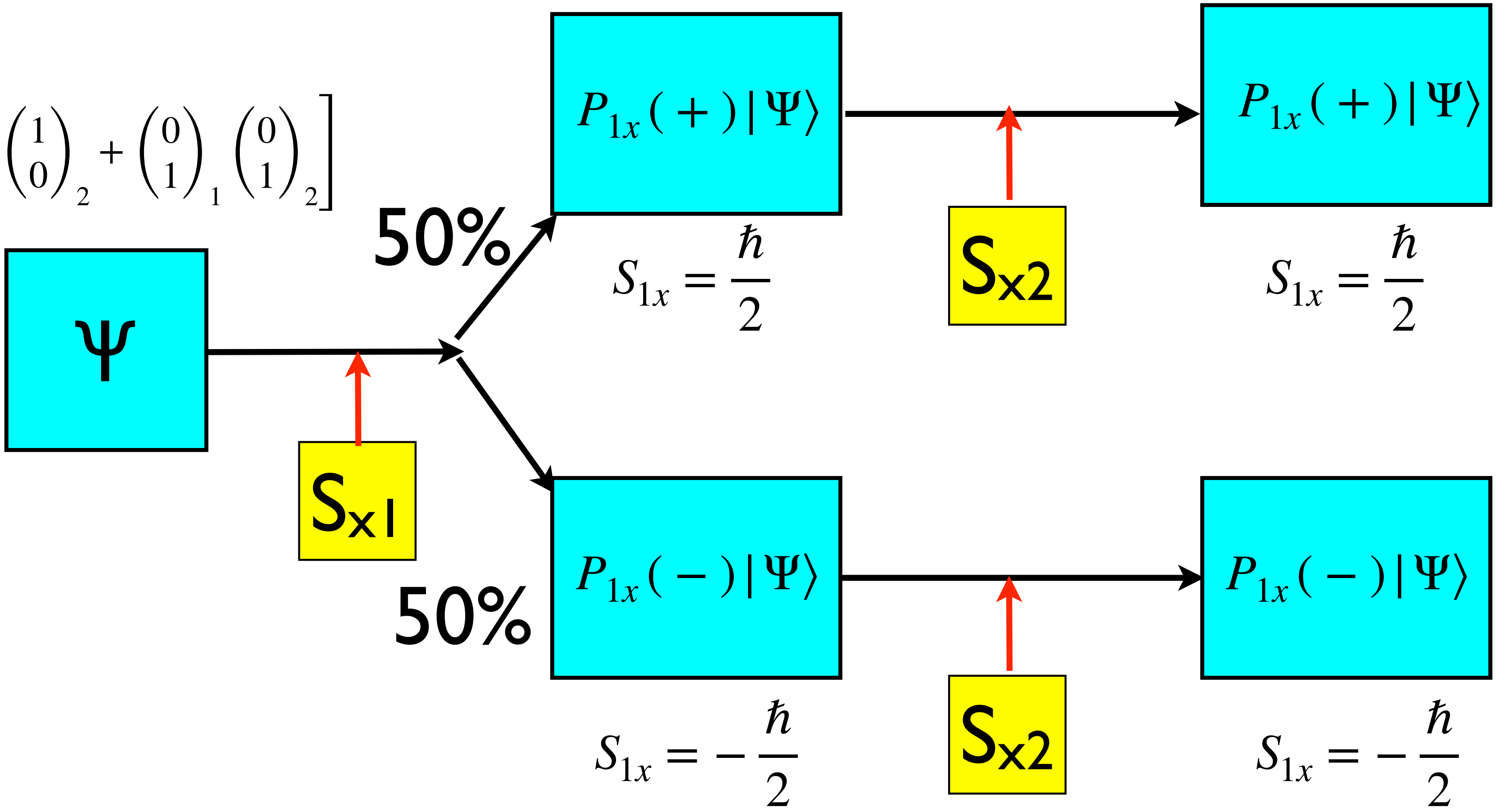
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}1\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_1\begin{pmatrix}1\\1\end{pmatrix}_2 = |\Psi'\rangle$$

- The following measurement on S_{x2} will only give `+’ result

$$S_{2x}|\Psi'\rangle = \frac{\hbar}{4\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_2\begin{pmatrix}1\\1\end{pmatrix}_1\begin{pmatrix}1\\1\end{pmatrix}_2 = \frac{\hbar}{2}|\Psi'\rangle$$

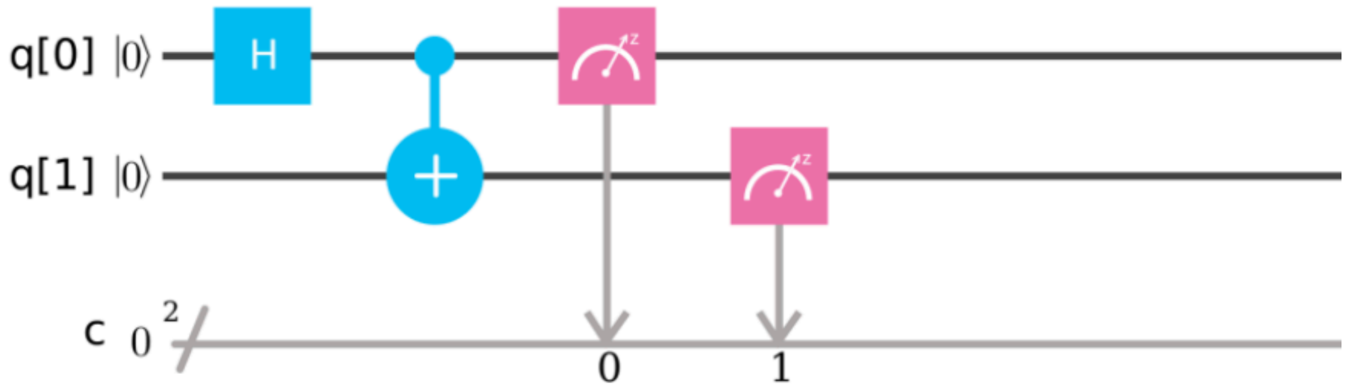
$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$



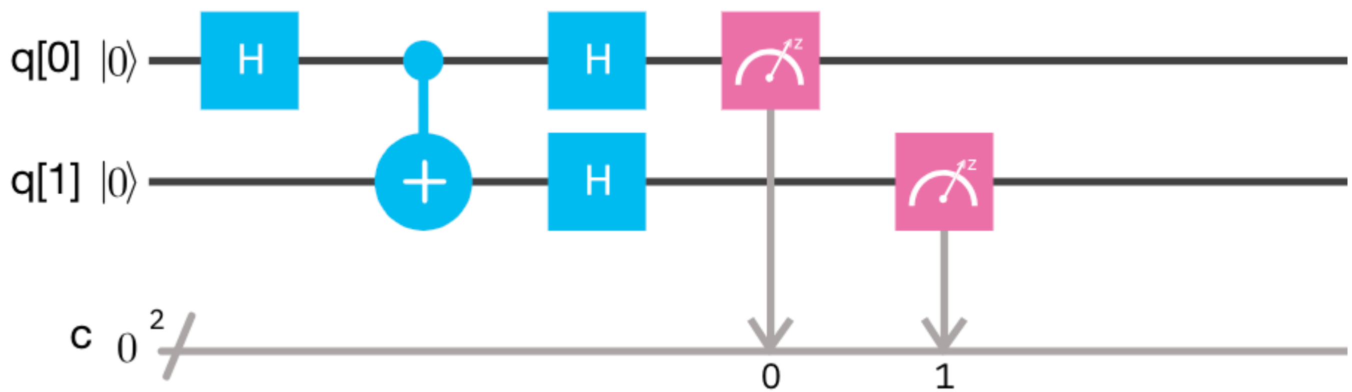
Bell state

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$$

Z-Z measurement

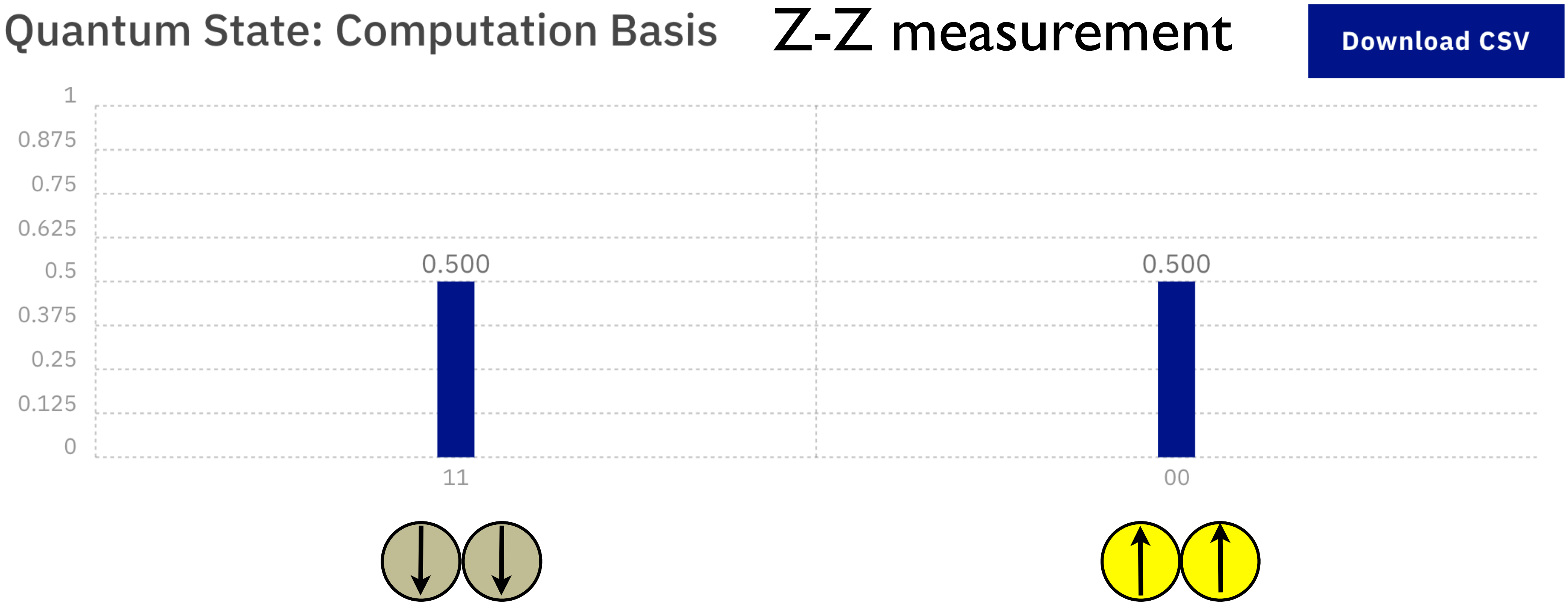


X-X measurement



Computation result

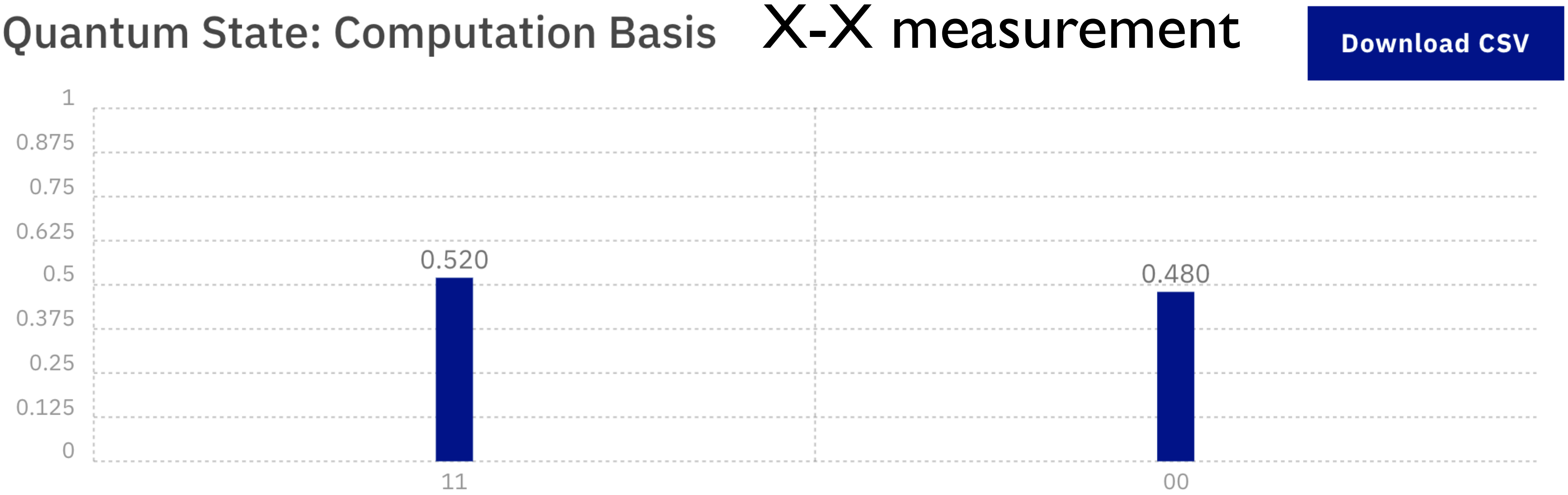
Z-Z measurement



All in -z direction

All in +z direction

Computation result



All in -x direction

All in +x direction

- Einstein's comment: "spukhafte Fernwirkung" or "spooky action at a distance