Symmetry

Symmetries and Unitary Transformations

- Let F and F' be two inertial frames
- Consider a system S prepared in arbitrary states $|\Psi_\alpha>$, $|\Psi_\beta>$,... to certain specifications by observables attached to F, and the states $|\Psi'\alpha>$, $|\Psi'_\beta>$ of S satisfying precisely the same specifications by observables attached to F'.

- On the assumption that these frames are equivalent, all the probabilities relating the states prepared in F must be equal to the corresponding relations in F':

$$\left|\left\langle \Psi_{\alpha} \middle| \Psi_{\beta} \right\rangle\right|^2 = \left|\left\langle \Psi'_{\alpha} \middle| \Psi'_{\beta} \right\rangle\right|^2$$

- the corresponding probability amplitudes are equal apart from a phase factor

$$\left\langle \Psi_{\alpha} \middle| \Psi_{\beta} \right\rangle = e^{i\lambda} \left\langle \Psi'_{\alpha} \middle| \Psi'_{\beta} \right\rangle$$

Spatial Translations

- the unitary operator for a spatial translation a is

$$T(a) = e^{-iP\cdot a/\hbar}$$

- where a is a numerical 3-vector, and P is the total momentum operator for the system in question

$$\left[P_i, P_j\right] = 0$$

- Let x_n be the coordinate operator of particle *n*.

$$T^{\dagger}(a) x_n T(a) = x_n + a$$

- If $|\varphi\rangle$ is any state, then

$$T(a)|\varphi\rangle = |\varphi; a\rangle$$

- let $F(x_n)$ be any observable constructed from coordinates. Then

$$T^\dagger(a) F(x_n) T(a) = F(x_n + a)$$

- for an infinitesimal translation,

$$F(x_n + a) = F(x_n) + \frac{i}{\hbar} \sum_i a_i \left[P_i, F(x_n) \right]$$

- if a function of the coordinates is invariant under a translation along the i-th direction, it commutes with that component of the total momentum.

- if the Hamiltonian is invariant under spatial translations, the total momentum commutes with the Hamiltonian and is therefore a constant of motion.

Groups of translation op

- Take the translation through *a* followed by *b*: $$T(b)T(a) = T(a+b)$$
- the order in these translations does not matter; they commute.
- The special case *b* = -*a*,

$$T(a)T(-a) = T(a)T^{\dagger}(a) = 1$$

- the operators T(a) form an Abelian Lie group of unitary operators standing in one-to-one correspondence with the group of translation in the Euclidean 3-space $\mathfrak{E}_3$.

- A group ($\mathfrak{G}$) is a finite or infinite set of elements ($g_1, g_2, \ldots$) having a composition law for every pair of elements such that $g_1 g_2$ is again an element of ($\mathfrak{G}$); which is associative, i.e., $(g_1 g_2)g_3 = g_1(g_2 g_3)$; and with every element g_i having an inverse g_i^{-1} such that $g_i g_i^{-1}$ is the identity element I, i.e., $Ig_i = g_i I = g_i$ for all i.
- A group is Abelian if all its elements commute, i.e., $g_1 g_2 = g_2 g_1$
- A group with an infinite set of elements is a Lie group if its elements can be uniquely specified by a set of continuous parameters $(z_1 \ldots z_r)$

Representation

- matrices whose multiplication law stands in one-to-one correspondence with the algebra of the group.

- if {|ξ>} is any basis in 𝔥, the matrices with elements $\langle \xi | T | \xi' \rangle$ form a representation of this group

$$\sum_{\xi''} \langle \xi | T(a) | \xi'' \rangle \langle \xi'' | T(b) | \xi' \rangle = \langle \xi | T(a+b) | \xi' \rangle$$

infinitesimal transformation

- the generalization of the infinitesimal translation

$$T = 1 - \frac{i}{\hbar}\delta a \cdot P$$

- if a unitary operator $U(z_l \ldots z_r)$ carries out a transformation belonging to a Lie group, then if the transformation is infinitesimal it has the form

$$U = 1 - i\sum_{l} \delta z_l \cdot \mathcal{G}_l$$

Generators

- the operators $\mathcal{G}_l$, which must be Hermitian for *U* to be unitary, are called the generators of the group ($\mathfrak{G}$).
- let $f(x_1 x_2 x_3)$ be any function of the coordinates in $\mathfrak{E}_3$, taken now to be real numbers and not operators, and consider the infinitesimal translation $x_i \to x_i + \delta a_i$

$$\delta f = f\left(x_i + \delta a_i\right) - f\left(x_i\right) = \sum_i \delta a_i \frac{\partial f}{\partial x_i}$$

$$\delta f = \frac{i}{\hbar} \sum_i \delta a_i \frac{\hbar}{i} \frac{\partial f}{\partial x_i}$$

Rotations

- Parametrization: specify a rotation *R* by the unit vector *n* along an axis of rotation, and an angle of rotation (θ) about that axis
- infinitesimal rotation will be parametrized by
$$\delta\omega = n\delta\theta$$
- Under this rotation, a vector K in $\mathfrak{E}_3$ transforms as follows:

$$K \to K + \delta K = K + \delta\omega \times K$$

$$\delta K_i = \varepsilon_{ijk}\delta\omega_j K_k$$

ε_{ijk} antisymmetric Levi-Civita tensor

Matrix representation

- If K is written as a column 3-vector, rotations through any angle can be carried out with the help of the following 3 x 3 matrices:

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad I_1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- A finite rotation of K about a single axis, the axis 1 through the angle φ_1, is by

$$K \rightarrow K' = Ke^{-i\phi_1 I_1}$$

non-Abelian group

- Successive rotations of K about distinct axes do not commute, a fact that is captured in the commutation rule

$$\left[I_i, I_j\right] = i\varepsilon_{ijk} I_k$$

- a unitary transformation D(R) on the Hilbert space S) of the system of interest.

$$D\left(R_2 R_1\right) = D\left(R_2\right) D\left(R_1\right)$$

$$D\left(R_2\right) D\left(R_1\right) \neq D\left(R_1\right) D\left(R_2\right)$$

angular momentum

- The general rotation can be expressed as

$$D(R) = \exp(-i\theta n \cdot J)$$

- n. J is the component of angular momentum along the direction n.

- considering two successive infinitesimal rotations about two distinct axes, say, through $\left(\delta\phi_1, \delta\phi_2\right)$

$$K' = \left(1 - i\delta\phi_2 I_2 + \cdots\right)\left(1 - i\delta\phi_1 I_1 + \cdots\right)K$$
$$K'' = \left(1 - i\delta\phi_1 I_1 + \cdots\right)\left(1 - i\delta\phi_2 I_2 + \cdots\right)K$$

$$K'' - K' = \left[-\delta\phi_1\delta\phi_2\left(I_1 I_2 - I_2 I_1\right) + \cdots\right]K \simeq -i\delta\phi_1\delta\phi_2 I_3$$

- The correspondence to the unitary operators *D*(*R*) must maintain this difference.

$$\begin{aligned} D\left(R_1 R_2\right) - D\left(R_2 R_1\right) &= e^{-i\delta\phi_1 J_1} e^{-i\delta\phi_2 J_2} - e^{-i\delta\phi_2 J_2} e^{-i\delta\phi_1 J_1} \\ &\simeq -\delta\phi_1\delta\phi_2\left(J_1 J_2 - J_2 J_1\right) \\ &= -i\delta\phi_1\delta\phi_2 J_3 \end{aligned}$$

commutation rule of angular momentum

- in general, the angular momentum commutation rule.

$$\left[J_i, J_j\right] = i\varepsilon_{ijk} J_k$$

- Let A be any observable. Under the rotation *R* it undergoes the unitary transformation

$$A \to A' = D^{\dagger}(R) A D(R)$$

$$A \to A' = A + \delta A$$

$$\delta A = i\delta\theta\left[n \cdot J, A\right]$$

- If an observable is invariant under rotations about *n*, it commutes with the corresponding component of angular momentum.
- An observable that is invariant under rotations about all directions is called a scalar under rotations.
- If the Hamiltonian is invariant under rotations about an axis, the component of the total angular momentum along that axis is a constant of motion.

vector operators

- a set of three observables $(V_1, V_2, V_3) = V$ is called a vector operator if it transforms under rotations in the same way as does the c-number vector K in

$$\delta V = i\delta\theta\left[n \cdot J, V\right] = \delta\theta n \times V$$

- V must therefore obey the commutation rule

$$\left[V_i, J_j\right] = i\varepsilon_{ijk} V$$

finite rotations

- consider a rotation about anyone axis.

$$V_i(\theta) = e^{i\theta J_3} V_i e^{-i\theta J_3}$$

$$\frac{dV_i(\theta)}{d\theta} = \dot{V}(\theta) = ie^{i\theta J_3}\left(J_3 V_i - V_i J_3\right)e^{-i\theta J_3} = ie^{i\theta J_3}\left[J_3, V_i\right]e^{-i\theta J_3}$$

$$\dot{V}_1(\theta) = ie^{i\theta J_3}\left[J_3, V_1\right]e^{-i\theta J_3} = e^{i\theta J_3} V_2 e^{-i\theta J_3} = -V_2(\theta)$$

$$\dot{V}_2(\theta) = ie^{i\theta J_3}\left[J_3, V_2\right]e^{-i\theta J_3} = V_1(\theta)$$

$$\dot{V}_3(\theta) = 0$$

- After integration

$$V_1(\theta) = V_1 \cos\theta - V_2 \sin\theta$$

$$V_2(\theta) = V_1 \sin\theta + V_2 \cos\theta$$

- The total momentum *P* of a system, being a vector, does not commute with the total angular momentum *J*; $\left[P_i, J_j\right] = i\varepsilon_{ijk} P_k$
- only components of P and J along the same direction commute. $\left[P_i, J_i\right] = 0$
- The scalar product of two vector operators is invariant under rotation, and must commute with *J*. J^2 and P^2 both commute with *J*,

$$\left[P^2, J_i\right] = 0 \qquad \left[J^2, J_i\right] = 0$$

- While all components of *P* and *J* are constants of motion if the Hamiltonian is invariant under translations and rotations, they cannot all be diagonalized simultaneously. Hence it is not possible to construct simultaneous eigenstates of all these 6 constants of motion.
- In addition to the rotational scalars P^2, J^2 and $P \cdot J$, one component of angular momentum, traditionally defined to be J_3, can be diagonalized simultaneously, and states can be designated by the associated eigenvalues.

Dimensionless angular momentum

- Consider a single particle with position and momentum operators *x* and *p*. The (dimensionless) orbital angular momentum operator *L* for this particle is then defined as

$$L = (x \times p)/\hbar$$

$$L_i = \varepsilon_{ijk} x_j p_k / \hbar$$

- the order of x_j and p_k does not matter because only commuting factors appear

$$[x_j, p_k] = 0 \quad \text{if } j \neq k$$

- The commutation rule for the orbital angular momentum

$$\left[L_i, L_j\right] = \frac{1}{\hbar^2}\left[\varepsilon_{ikl} x_k p_l, \varepsilon_{jmn} x_m p_n\right] = \frac{\varepsilon_{ikl}\varepsilon_{jmn}}{\hbar^2}\left[x_k p_l, x_m p_n\right]$$

$$\begin{aligned}\left[x_k p_l, x_m p_n\right] &= \left[x_k, x_m p_n\right] p_l + x_k\left[p_l, x_m p_n\right] \\ &= x_m\left[x_k, p_n\right] p_l + x_k\left[p_l, x_m\right] p_n \\ &= i\hbar\left(\delta_{kn} x_m p_l - \delta_{lm} x_k p_n\right)\end{aligned}$$

$$
\begin{aligned}
\left[L_i, L_j\right] &= \frac{i}{\hbar}\left(\varepsilon_{ikl}\varepsilon_{jmk}x_m p_l - \varepsilon_{ikl}\varepsilon_{jln}x_k p_n\right) \\
&= \frac{i}{\hbar}\left[\left(\delta_{jl}\delta_{im} - \delta_{ij}\delta_{lm}\right)x_m p_l - \left(\delta_{in}\delta_{jk} - \delta_{ij}\delta_{kn}\right)x_k p_n\right] \\
&= \frac{i}{\hbar}\left(x_i p_j - x_j p_i\right) - \frac{i}{\hbar}\delta_{ij}\left(x_l p_l - x_k p_k\right) \\
&= \frac{i}{\hbar}\left(x_i p_j - x_j p_i\right) \\
&= i\varepsilon_{ijk}L_k
\end{aligned}
$$

$$
\varepsilon_{ikl}\varepsilon_{imn} = \delta_{km}\delta_{nl} - \delta_{kn}\delta_{lm}
$$

- From the canonical commutation rules

$$\left[x_i, L_j\right] = \frac{1}{\hbar}\left[x_i, \varepsilon_{jkl} x_k p_l\right] = \frac{\varepsilon_{jkl}}{\hbar} x_k \left[x_i, p_l\right] = i\varepsilon_{jkl}\delta_{il} x_k = i\varepsilon_{ijk} x_k$$

$$\left[p_i, L_j\right] = \frac{1}{\hbar}\left[p_i, \varepsilon_{jkl} x_k p_l\right] = \frac{\varepsilon_{jkl}}{\hbar}\left[p_i, x_k\right] p_l = -i\varepsilon_{jkl}\delta_{ik} p_l = i\varepsilon_{ijl} p_l$$

Generator

- Let ψ(r) be some wave function, where r = (r_1, r_2, r_3) is the eigenvalue of x. Under a infinitesimal rotation about n = (0,0,1), the change in ψ is

$$
\begin{aligned}
\delta\psi(r) &= \psi(r_1 - r_2\delta\theta, r_2 + r_1\delta\theta, r_3) - \psi(r_1, r_2, r_3) \\
&= \delta\theta\left(-r_2\frac{\partial\psi}{\partial r_1} + r_1\frac{\partial\psi}{\partial r_2}\right) \\
&= \frac{i\delta\theta}{\hbar}(x_1 p_2 - x_2 p_1)\psi \\
&= i\delta\theta L_3 \psi
\end{aligned}
$$