Angular momentum

2017/3/30

central force problem

- in 3D system

$$H = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r})$$

- μ reduced mass
- central force, potential only depends on distance

$$V(\mathbf{r}) = V(r)$$

conservation of L

- Angular momentum is conserved in central force problem

$$\frac{d\mathbf{L}}{dt} = 0$$

- In quantum mechanics, it can be written as

$$[H, \mathbf{L}] = 0$$

angular mom operator

- in classical mechanics, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- all the 3 components are

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

- because the operators are commute, the order can be changed

commutation relations

- mutual commutation relations

$$\left[L_x, L_y\right] = \left[yp_z - zp_y, zp_x - xp_z\right] = \left[yp_z, zp_x\right] + \left[zp_y, xp_z\right]$$

$$= -i\hbar y p_x + i\hbar p_y x = i\hbar L_z$$

$$\left[L_y, L_z\right] = i\hbar L_x$$

$$\left[L_z, L_x\right] = i\hbar L_y$$

- L^2

$$\left[L_x, L^2\right] = \left[L_x, L_x^2 + L_y^2 + L_z^2\right] = \left[L_x, L_y^2\right] + \left[L_x, L_z^2\right]$$

$$= i\hbar\left(L_y L_z + L_z L_y\right) - i\hbar\left(L_y L_z + L_z L_y\right) = 0$$

$$\left[L_y, L^2\right] = 0$$

$$\left[L_z, L^2\right] = 0$$

mutual eigenstates

- the components are not commute, they do not have common eigenstates.
- We choose simultaneous eigenstates of L_z and L^2 as $|l,m\rangle$
- eigenvalues

$$L_z|l,m\rangle = m\hbar|l,m\rangle$$
$$L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$$

why? we will see later

raising and lowering operators

- to show the structure, we define

$$L_{\pm} = L_x \pm iL_y$$

- commutation relations

$$\left[L_+, L_-\right] = \left[L_x + iL_y, L_x - iL_y\right] = -2i\left[L_x, L_y\right] = 2\hbar L_z$$

$$\left[L_z, L_+\right] = \left[L_z, L_x + iL_y\right] = i\hbar L_y + \hbar L_x = \hbar L_+$$

$$\left[L_z, L_-\right] = \left[L_z, L_x - iL_y\right] = i\hbar L_y - \hbar L_x = -\hbar L_-$$

$$\left[L^2, L_{\pm}\right] = \left[L^2, L_x \pm iL_y\right] = 0$$

some properties

$$L_+L_- = \left(L_x + iL_y\right)\left(L_x - iL_y\right)$$

$$= L_x^2 + L_y^2 + i\left[L_y, L_x\right]$$

$$= L_x^2 + L_y^2 + \hbar L_z$$

$$L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$= L_+L_- - \hbar L_z + L_z^2$$

$$= L_-L_+ + \hbar L_z + L_z^2$$

meaning of L_+

- What is $L_+|l,m\rangle$
- we check the length

$$L^2 L_+|l,m\rangle = L_+ L^2|l,m\rangle = l(l+1) L_+|l,m\rangle$$

- we check the z-component

$$L_z L_+|l,m\rangle = \left(L_+ L_z + \hbar L_+\right)|l,m\rangle = (m+1)\hbar L_+|l,m\rangle$$

- the length is unchanged and z-component eigenvalue is added by 1

$$L_+|l,m\rangle = C_+|l,m+1\rangle$$

meaning of L-

- Similarly,

$$L^2 L_- |l,m\rangle = l(l+1) L_- |l,m\rangle$$

$$L_z L_- |l,m\rangle = (m-1)\hbar L_- |l,m\rangle$$

$$L_- |l,m\rangle = C_- |l,m-1\rangle$$

coefficients

- to calculate *C*

$$\langle l,m|L_+^\dagger L_+|l,m\rangle = \langle l,m|L_-L_+|l,m\rangle$$

$$= \langle l,m|L^2 - L_z^2 - \hbar L_z|l,m\rangle$$

$$= l(l+1)\hbar^2 - m^2\hbar^2 - m\hbar^2$$

$$= (l-m)(l+m+1)\hbar^2 = |C_+|^2$$

$$\langle l,m|L_-^\dagger L_-|l,m\rangle = \langle l,m|L_+L_-|l,m\rangle$$

$$= (l+m)(l-m+1)\hbar^2 = |C_-|^2$$

$$L_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l,m+1\rangle$$

$$L_-|l,m\rangle = \sqrt{(l+m)(l-m+1)}\hbar|l,m-1\rangle$$

possible values of *m*

- the coefficients should be

$$\left|C_\pm\right|^2 \geq 0$$

- it gives a constraint for *m*

$$(l-m)(l+m+1) \geq 0$$
$$(l+m)(l-m+1) \geq 0$$

$$-l \leq m \leq l$$

structure of *l* and *m*

- starting from a maximal $m=l$, we can lower the state to

$$m = l, l-1, l-2\cdots$$

- it should stop at the minimal value $m=-l$

$$m = l, l-1, l-2\cdots, -l+1, -l$$

- The total number of possible m is $(2l+1)$
- possible value of l: integers and half-integers

representation in θ, φ

- The integer angular momentum states can be expressed using spherical coordinate (θ, φ)
- We may show that the spherical harmonics functions are the eigenstates

$$Y_{lm}(\theta,\phi) = \langle \theta,\phi | l,m \rangle$$

angular momentum

- the operators in spherical coordinate
- We express the components of angular momentum to be

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = x^2 + y^2 + z^2$$

$$\cos\theta = \frac{z}{r}$$

$$\tan\phi = \frac{y}{x}$$

calculation of Jacobian

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \cos\theta}{\partial x}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial x}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial x} = \frac{xz}{r^3\sin\theta} = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \cos\theta}{\partial y}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial y}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial y} = \frac{yz}{r^3\sin\theta} = \frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial \theta}{\partial z} = \frac{\partial \cos\theta}{\partial z}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = -\frac{1}{r\sin\theta} + \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial z} = -\frac{x^2+y^2}{r^3\sin\theta} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \tan\phi}{\partial x}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\cos^2\phi\frac{y}{x^2} = -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \tan\phi}{\partial y}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \cos^2\phi\frac{1}{x} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial z} = 0$$

L operators in θ, φ

$$\frac{i}{\hbar}L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$= y\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) - z\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)$$

$$= y\left(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - z\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \cos^2\phi\frac{1}{x}\frac{\partial}{\partial\phi}\right)$$

$$= -\left(y\frac{\sin\theta}{r} + z\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} - \cos^2\phi\frac{z}{x}\frac{\partial}{\partial\phi}$$

$$= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\frac{i}{\hbar}L_y = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

all these calculations can be done by sympy

raising and lowering op

$$
\begin{aligned}
L_{\pm} &= L_x \pm iL_y \\
&= i\hbar \sin\phi \frac{\partial}{\partial\theta} + i\hbar \cot\theta \cos\phi \frac{\partial}{\partial\phi} \pm \hbar\left(\cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right) \\
&= \pm\hbar e^{\pm i\phi} \frac{\partial}{\partial\theta} + i\hbar \cot\theta e^{\pm i\phi} \frac{\partial}{\partial\phi} \\
&= \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right)
\end{aligned}
$$

- find φ solution

$$\langle\theta,\phi|L_z|l,m\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\langle\theta,\phi|l,m\rangle = m\hbar\langle\theta,\phi|l,m\rangle$$

$$\frac{\partial}{\partial\phi}\langle\theta,\phi|l,m\rangle = im\langle\theta,\phi|l,m\rangle$$

$$\langle\theta,\phi|l,m\rangle = e^{im\phi}F(\theta)$$

- find θ solution $\quad L_+|l,l\rangle = 0$

$$\langle \theta,\phi | L_+ | l,l\rangle = -\hbar e^{i\phi}\frac{\partial}{\partial\theta}\langle \theta,\phi | l,l\rangle - i\hbar\cot\theta e^{i\phi}\frac{\partial}{\partial\phi}\langle \theta,\phi | l,l\rangle = 0$$

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\langle \theta,\phi | l,l\rangle = 0$$

$$\left(\frac{\partial}{\partial\theta} - l\cot\theta\right)F(\theta) = 0$$

$$\left(\frac{\partial}{\cos\theta\,\partial\theta} - \frac{l}{\sin\theta}\right)F(\theta) = 0$$

$$F(\theta) = \sin^l\theta$$

$$Y_{lm}(\theta,\phi) = \langle \theta,\phi | l,m \rangle$$

$$= C\langle \theta,\phi | (L_-)^{l-m} | l,l \rangle = C\hbar^{l-m} \left[e^{-i\phi} \left(\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\phi} \right) \right]^{l-m} \sin^l \theta e^{il\phi}$$

- normalization constant $\int \left| \langle \theta,\phi | l,m \rangle \right|^2 d\Omega = 1$

$$Y_{lm}(\theta,\phi) = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta) e^{im\phi}$$

matrix representation

- for angular momentum l=1, there are 3 eigenstates
- we can write them with column vectors

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{matrix} 1 \\ 0 \\ -1 \end{matrix}$$

row index

matrix elements

- matrix representation of L operators

$$\langle l,m' | L_z | l,m \rangle = m\hbar\delta_{m',m}$$

$$\langle l,m' | L_+ | l,m \rangle = \sqrt{(l-m)(l+m+1)}\hbar\delta_{m',m+1}$$

$$\langle l,m' | L_- | l,m \rangle = \sqrt{(l+m)(l-m+1)}\hbar\delta_{m',m-1}$$

row index   column index

$$L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

1 0 -1

$$L_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

1
0
-1

matrix elements

- for other L components

$$L_x = \frac{1}{2}\left(L_+ + L_-\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = -\frac{i}{2}\left(L_+ - L_-\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

- Easy to tell L_+ and L_- are non-Hermitian, and L_i are Hermitian

Commutation relations

- It is also easy to prove commutation relations using matrix representation

$$\left[L_x, L_y\right] = L_x L_y - L_y L_x = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\hbar^2}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \frac{\hbar^2}{2}\begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} = i\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= i\hbar L_z$$

matrix representation for SHO

- The operators in SHO

$$H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

- the eigenstates in SHO are written as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

matrix representation

$$\langle n'|H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)\delta_{n',n}$$

$$H = \hbar\omega\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{2} & 0 \\ & & & & \ddots \end{pmatrix}$$

matrix representation

$$\langle n' | a | n \rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ & & & & \ddots \end{pmatrix}$$

$$\langle n' | a^\dagger | n \rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ & & & & \ddots \end{pmatrix}$$