Charged particles in EM fields

2018/5/15

Classical Electrodynamics

- Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

potentials and gauge transformation

- The scalar and vector potentials are defined by

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

$$\left(\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \right)$$

- Fields are invariant under the gauge transformation on potentials

$$\mathbf{A}' = \mathbf{A} - \nabla g$$

$$\phi' = \phi + \frac{\partial g}{\partial t}$$

Maxwell equations for potentials

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$-\nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) = \frac{\rho}{\varepsilon_0}$$

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{A} \right) = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) = \mu_0 \mathbf{j} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right)$$

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \mu_0 \mathbf{j}$$

Coulomb gauge

- if charge distribution is static

$$\nabla \cdot \mathbf{A} = 0$$

- And the Maxwell equations becomes

$$-\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$$

- The scalar potential is time independent $\frac{\partial \phi}{\partial t} = 0$

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

Lorentz gauge

- For nonstatic charge distribution, we may choose

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

- We may get the same equation for *A*

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

- and a wave equation for φ

$$-\nabla^2 \phi + \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}$$

Hamilton formalism

- In absence of EM fields, the particle hamiltonian is

$$H = \frac{p^2}{2m} + V(r)$$

- The equations of motion

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

$$\frac{dx_i}{dt} = \frac{p_i}{m}$$

$$\frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i}$$

- Together we may have

$$m\frac{d^2 x_i}{dt^2} = \frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i}$$

Lorentz force

- The particle experiences a force from the EM fields(eg electron)

$$m_e \frac{d^2\mathbf{r}}{dt^2} = -e\mathbf{E} + \mathbf{v} \times \mathbf{B}$$

- We could take account in its contribution by considering the hamiltonian

$$H = \frac{\left(\mathbf{p} + e\mathbf{A}\right)^2}{2m_e} - e\phi$$

equations of motion

- The eqs of motion

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m_e}\left(p_i + eA_i\right)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{e}{m_e}\sum_k \left(p_k + eA_k\right)\frac{\partial A_k}{\partial x_i} + e\frac{\partial \phi}{\partial x_i}$$

- combine together:

$$m_e\frac{d^2x_i}{dt^2} = \frac{d}{dt}\left(p_i + eA_i\right) = -\frac{e}{m_e}\left(p_k + eA_k\right)\frac{\partial A_k}{\partial x_i} + e\frac{\partial \phi}{\partial x_i} + \frac{d}{dt}eA_i$$

$$= -\frac{e}{m_e}\left(m_e\frac{dx_k}{dt}\right)\frac{\partial A_k}{\partial x_i} + e\frac{\partial \phi}{\partial x_i} + e\frac{\partial A_i}{\partial t} + e\frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt}$$

$$= -e\frac{dx_k}{dt}\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right) + e\left(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial t}\right)$$

$$= -e\left(\mathbf{v}\times\mathbf{B}\right)_i - e\mathbf{E}_i$$

Schrodinger equation

- Hamiltonian $$H = \frac{\left(\mathbf{p} + e\mathbf{A}\right)^2}{2m_e} - e\phi$$

- replace $$\mathbf{p} = \frac{\hbar}{i}\nabla$$

- Schrodinger eq $$i\hbar\frac{\partial\psi}{\partial t} = H\psi = \left[\frac{1}{2m_e}\left(-i\hbar\nabla + e\mathbf{A}\right)^2 - e\phi\right]\psi$$

gauge transformation

- Rewrite the Schrodinger eqn

$$\mathbf{A}' = \mathbf{A} - \nabla g$$

$$\phi' = \phi + \frac{\partial g}{\partial t}$$

$$i\hbar\frac{\partial \psi}{\partial t} = H\psi = \left[\frac{1}{2m_e}\left(-i\hbar\nabla + e\mathbf{A}' + e\nabla g\right)^2 - e\phi' - e\frac{\partial g}{\partial t}\right]\psi$$

- The solution is $\psi' = e^{i\Lambda}\psi$ $\quad i\hbar\frac{\partial \psi'}{\partial t} = H\psi'$

- The gauge transformation produces an extra phase to the wavefunction

gauge transformation

- The LHS $$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial}{\partial t}\left(e^{-i\Lambda}\psi'\right) = \hbar e^{-i\Lambda}\frac{\partial\Lambda}{\partial t}\psi' + i\hbar e^{-i\Lambda}\frac{\partial\psi'}{\partial t}$$

$$\psi = e^{-i\Lambda}\psi'$$

- The RHS $$-i\hbar\nabla\left(e^{-i\Lambda}\psi'\right) = -\hbar e^{-i\Lambda}\left(\nabla\Lambda\right)\psi' - i\hbar e^{-i\Lambda}\nabla\psi'$$

$$e^{-i\Lambda}\left[\frac{1}{2m_e}\left(-i\hbar\nabla + e\mathbf{A}' + e\nabla g - \hbar\nabla\Lambda\right)^2 - e\phi' - e\frac{\partial g}{\partial t}\right]\psi$$

- if choose $\Lambda = \frac{e}{\hbar}g$ we have

$$i\hbar e^{-i\Lambda}\frac{\partial\psi'}{\partial t} = e^{-i\Lambda}\left[\frac{1}{2m_e}\left(-i\hbar\nabla + e\mathbf{A}'\right)^2 - e\phi'\right]\psi \qquad \psi' = e^{i\frac{e}{\hbar}g}\psi$$

Coulomb gauge

- Schrodinger eq $\nabla \cdot \mathbf{A} = 0$

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{1}{2m_e} \left(-i\hbar \nabla + e\mathbf{A} \right)^2 - e\phi \right] \psi$$

$$= \frac{1}{2m_e} \left(-\hbar^2 \nabla^2 \psi - 2ie\hbar \left(\nabla \cdot \mathbf{A} \right) \psi - 2i\hbar e \mathbf{A} \cdot \nabla \psi + e^2 \mathbf{A}^2 \psi \right) - e\phi\psi$$

$$= -\frac{\hbar^2}{2m_e} \nabla^2 \psi - \frac{ie\hbar}{m_e} \mathbf{A} \cdot \nabla \psi + \frac{e^2 \mathbf{A}^2}{2m_e} \psi - e\phi\psi$$

- time independent equation $\psi\left(\mathbf{r}, t\right) = e^{\frac{i}{\hbar} Et} \psi\left(\mathbf{r}\right)$

$$-\frac{\hbar^2}{2m_e} \nabla^2 \psi - \frac{ie\hbar}{m_e} \mathbf{A} \cdot \nabla \psi + \frac{e^2 \mathbf{A}^2}{2m_e} \psi - e\phi\psi = E\psi$$

Constant magnetic field

- For a constant magnetic field B $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$
- The Schrodinger equation

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi - \frac{ie\hbar}{m_e}\mathbf{A}\cdot\nabla\psi + \frac{e^2\mathbf{A}^2}{2m_e}\psi - e\phi\psi = E\psi$$

- 2nd term

$$\left(-\frac{ie\hbar}{m_e}\right)\mathbf{A}\cdot\nabla\psi = \left(-\frac{ie\hbar}{m_e}\right)\left(-\frac{1}{2}\mathbf{r}\times\mathbf{B}\right)\cdot\nabla\psi = -\left(\frac{ie\hbar}{2m_e}\right)\mathbf{B}\cdot\left(\mathbf{r}\times\nabla\psi\right) = \frac{e}{2m_e}\mathbf{B}\cdot\mathbf{L}\psi$$

- 3rd term $\frac{e^2\mathbf{A}^2}{2m_e}\psi = \frac{e^2}{8m_e}\left(\mathbf{r}\times\mathbf{B}\right)^2\psi = \frac{e^2}{8m_e}\left[\left(rB\right)^2 - \left(\mathbf{r}\cdot\mathbf{B}\right)^2\right]\psi$

Harmonic trap

- choose the magnetic field direction to be z

$$\mathbf{A} = \left(-\frac{yB}{2}, \frac{xB}{2}, 0 \right)$$

- 2nd term $\frac{eB}{2m_e} L_z \psi$

- 3rd term $\frac{e^2}{8m_e}\left[(rB)^2 - (\mathbf{r}\cdot\mathbf{B})^2 \right]\psi = \frac{e^2 B^2}{8m_e}\left(x^2 + y^2 \right)\psi$

- It behaves like a 2D harmonic oscillator

- Contribution from angular momentum

$$\frac{eB}{2m_e}\left\langle L_z \right\rangle \sim \frac{eB}{2m_e}\hbar$$

- Contribution from harmonic trap for a bounded electron in a atom

$$\frac{e^2 B^2}{8m_e}\left\langle x^2 + y^2 \right\rangle \sim \frac{e^2 B^2}{8m_e} a_0^2$$

- ratio

$$\frac{\frac{e^2 B^2}{8m_e}\left\langle x^2 + y^2 \right\rangle}{\frac{eB}{2m_e}\left\langle L_z \right\rangle} \sim \frac{eBa_0^2}{4\hbar} \sim \frac{B}{10^6\,\mathrm{T}}$$

magnetic field produce small effect on atomic size

Particle in a harmonic trap

- In cylindrical coordinate, write the wavefucntion as

$$\psi = u(\rho) e^{im\phi} e^{ikz} \qquad \frac{eB}{2m_e} L_z \psi = \frac{eB}{2m_e} m\hbar\psi$$

- The Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial}{\partial \phi}$$

- The radial equation

$$-\frac{\hbar^2}{2m_e}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)u(\rho) + \frac{m_e\omega^2}{8}\rho^2 u(\rho) = \left(E - \frac{m\hbar\omega}{2} - \frac{\hbar^2 k^2}{2m_e}\right)u(\rho)$$

$\omega = \frac{eB}{m_e}$ called cyclotron frequency

Solve radial function

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)u(\rho)-\frac{m_e^2\omega^2}{4\hbar^2}\rho^2u(\rho)=-\frac{2m_e}{\hbar^2}\left(E-\frac{m\hbar\omega}{2}-\frac{\hbar^2k^2}{2m_e}\right)u(\rho)$$

- Again, do scaling $\sqrt{\frac{m_e\omega}{2\hbar}}\rho = x = \sqrt{\frac{eB}{2\hbar}}\rho$
- eigenequation

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}-x^2+\lambda\right)u(x)=0$$

- dimensionless eigenvalue

$$\lambda=\frac{4}{\hbar\omega}\left(E-\frac{m\hbar\omega}{2}-\frac{\hbar^2k^2}{2m_e}\right)=\frac{4}{\hbar\omega}\left(E-\frac{\hbar^2k^2}{2m_e}\right)-2m$$

Asymptotic behavior

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}-x^2+\lambda\right)u(x)=0$$

- large *x*

$$\frac{d^2u}{dx^2}-x^2u\sim 0$$

$$u\sim e^{-\frac{x^2}{2}}$$

- small *x*

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}-\frac{m^2}{x^2}u=0$$

$$u\sim x^{|m|}$$

- combine together $u(x)=x^{|m|}e^{-\frac{x^2}{2}}G(x)$

Equation for G

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{m^2}{x^2}-x^2+\lambda\right)x^{|m|}e^{-\frac{x^2}{2}}G(x)=0$$

$$\frac{d}{dx}\left(x^{|m|}e^{-\frac{x^2}{2}}G\right)=|m|x^{|m|-1}e^{-\frac{x^2}{2}}G-x^{|m|+1}e^{-\frac{x^2}{2}}G+x^{|m|}e^{-\frac{x^2}{2}}G'=e^{-\frac{x^2}{2}}x^{|m|}\left(|m|x^{-1}G-xG+G'\right)$$

$$\frac{d^2}{dx^2}\left(x^{|m|}e^{-\frac{x^2}{2}}G\right)=|m|\left(|m|-1\right)x^{|m|-2}e^{-\frac{x^2}{2}}G+x^{|m|+2}e^{-\frac{x^2}{2}}G+x^{|m|}e^{-\frac{x^2}{2}}G''$$

$$-\left(2|m|+1\right)x^{|m|}e^{-\frac{x^2}{2}}G+2|m|x^{|m|-1}e^{-\frac{x^2}{2}}G'-2x^{|m|+1}e^{-\frac{x^2}{2}}G'$$

$$=e^{-\frac{x^2}{2}}x^{|m|}\left[|m|\left(|m|-1\right)x^{-2}G+x^2G+G''-\left(2|m|+1\right)G+2|m|x^{-1}G'-2xG'\right]$$

$$e^{-\frac{x^2}{2}}x^{|m|}\left[|m|\left(|m|-1\right)x^{-2}G+x^2G+G''-\left(2|m|+1\right)G+2|m|x^{-1}G'-2xG'\right]$$

$$+e^{-\frac{x^2}{2}}x^{|m|}\left(|m|x^{-2}G-G+x^{-1}G'\right)+e^{-\frac{x^2}{2}}x^{|m|}\left(-|m|^2x^{-2}G-x^2G+\lambda G\right)=0$$

$$G''+\left(2|m|+1\right)x^{-1}G'-2xG'+\left(\lambda-2|m|-2\right)G=0$$

change variable

- change variable $y = x^2$

$$\frac{d}{dx} = 2x\frac{d}{dy}$$

$$\frac{d^2}{dx^2} = 2x\frac{d}{dy}\left(2x\frac{d}{dy}\right) = 4x^2\frac{d^2}{dy^2} + 2\frac{d}{dy}$$

$$G'' + \left(2|m|+1\right)x^{-1}G' - 2xG' + \left(\lambda - 2|m| - 2\right)G$$

$$4x^2\frac{d^2G}{dy^2} + 2\frac{dG}{dy} + \left[\left(2|m|+1\right)x^{-1} - 2x\right]2x\frac{dG}{dy} + \left(\lambda - 2|m| - 2\right)G = 0$$

$$2y\frac{d^2G}{dy^2} + \frac{dG}{dy} + \left[\left(2|m|+1\right) - 2y\right]\frac{dG}{dy} + \frac{\left(\lambda - 2|m| - 2\right)}{2}G = 0$$

$$\frac{d^2G}{dy^2} + \left[\frac{|m|+1}{y} - 1\right]\frac{dG}{dy} + \frac{\left(\lambda - 2|m| - 2\right)}{4y}G = 0$$

eigenvalue

- Recall the radial function in hydrogen atom problem

$$\frac{\partial^2 H}{\partial \rho^2} + \left(\frac{2l+2}{\rho} - 1\right)\frac{\partial H}{\partial \rho} + \frac{\lambda - l - 1}{\rho} H = 0$$

- eigenvalues

$$\lambda = k + l + 1 \qquad k = 0, 1, \cdots \qquad H(\rho) = L_k^{(2l+1)}(\rho)$$

- In this problem $$\frac{d^2 G}{dy^2} + \left(\frac{|m|+1}{y} - 1\right)\frac{dG}{dy} + \frac{(\lambda - 2|m| - 2)}{4y} G = 0$$

$$\frac{\lambda}{4} - \frac{|m|+1}{2} = 0, 1, \cdots = n \qquad G(y) = L_n^{|m|}(y)$$

eigenenergy

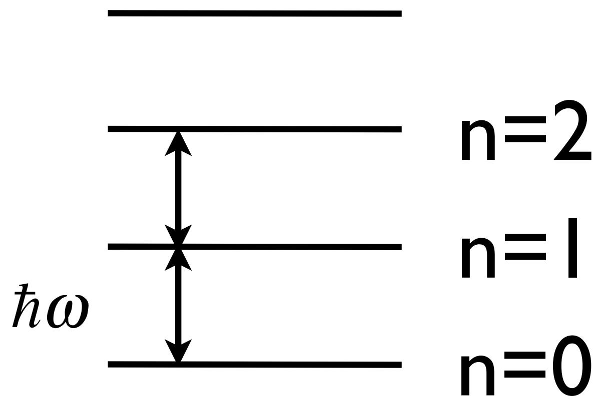
$$\frac{\lambda}{4} - \frac{|m|+1}{2} = n_r \qquad \lambda = 4n_r + 2|m| + 2$$

$$E - \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar\omega}{4}\left(\lambda - 2m\right) = \frac{\hbar\omega}{2}\left(2n_r - m + |m| + 1\right)$$

- equally spaced energy levels

$$E - \frac{\hbar^2 k^2}{2m_e} = \hbar\omega\left(n_r + \frac{1}{2}\right) \quad \text{if} \quad m > 0$$

$$= \hbar\omega\left(n_r + m + \frac{1}{2}\right) \quad \text{if} \quad m < 0$$



Classical cyclotron motion

- In classical mechanics

$$H = \frac{\left(\mathbf{p} + e\mathbf{A}\right)^2}{2m_e} \qquad v_i = \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m_e}\left(p_i + eA_i\right)$$

- angular momentum

$$m_e \mathbf{r} \times \mathbf{v} = m_e \mathbf{r} \times \frac{\mathbf{p} + e\mathbf{A}}{m_e} = \mathbf{r} \times \mathbf{p} + e\mathbf{r} \times \mathbf{A} \qquad \mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$$

$$= \mathbf{L} - \frac{e}{2}\mathbf{r} \times \left(\mathbf{r} \times \mathbf{B}\right) = \mathbf{L} - \frac{e}{2}\left(\left(\mathbf{r} \cdot \mathbf{B}\right)\mathbf{r} - r^2\mathbf{B}\right)$$

- the z-component

$$m_e \rho v = L_z - \frac{e}{2}\rho^2 B$$

Classical cyclotron motion

- Force balance $$m_e \frac{v^2}{\rho} = evB$$

$$m_e v = eB\rho$$

$$m_e \rho v = L_z - \frac{eB}{2}\rho^2$$

$$L_z = \frac{eB\rho^2}{2}$$

$$\rho = \sqrt{\frac{2L_z}{eB}}$$

- Kinetic energy

$$E = \frac{1}{2} m_e v^2 = \frac{e^2 B^2 \rho^2}{2m_e} = \frac{eB}{m_e} L_z$$

- In quantum theory when *m* is large

$$L_z = m\hbar$$

$$E - \frac{\hbar^2 k^2}{2m_e} \sim m\hbar\omega = \frac{eB}{m_e} L_z$$

Orbit radius

- The fundamental mode

$$G(y) = L_0^{|m|}(y) = \text{constant}$$

- The radial function $u \sim x^{|m|} e^{-\frac{x^2}{2}}$
- probability $P \sim x^{2|m|} e^{-x^2}$
- most probable position

$$\frac{dP}{dx} \sim 2\left(|m| x^{-1} - x\right) x^{2|m|} e^{-x^2} = 0$$

$$x = \sqrt{|m|}$$

$$\rho = \sqrt{\frac{2\hbar m}{eB}} = \sqrt{\frac{2L_z}{eB}}$$

Landau levels

- If we chose a different gauge $\mathbf{A} = (0, Bx, 0)$
- The kinetic energy

$$\frac{(\mathbf{p}+e\mathbf{A})^2}{2m_e} = \frac{1}{2m_e}\left(p_x^2 + (p_y + eBx)^2 + p_z^2\right)$$

note $[x, p_y] = 0$

$$= \frac{1}{2m_e}\left(p_x^2 + p_y^2 + p_z^2 + 2eBxp_y + e^2B^2x^2\right)$$

- The commutation relations

$$[H, p_y] = [H, p_z] = [p_y, p_z] = 0$$

- the energy eigenstates are simultanously engien states of p_y and p_z

Solution

- For simplicity, assume that k_z=0
- the wavefunction is written as

$$\psi = e^{iky} v(x)$$

- insert into Schrodinger equation

$$\frac{(\mathbf{p} + e\mathbf{A})^2}{2m_e}\psi = \frac{1}{2m_e} e^{iky}\left(-\hbar^2 \frac{d^2}{dx^2} + (\hbar k + eBx)^2\right) v(x)$$

$$\frac{1}{2m_e}\left(-\hbar^2 \frac{d^2}{dx^2} + e^2 B^2 \left(x + \frac{\hbar k}{eB}\right)^2\right) v(x) = Ev(x)$$

- A simple harmonic oscillator in *x* direction

1D harmonic trap

- The harmonic trap is centered at x_0

$$x_0 = \frac{\hbar k}{eB}$$

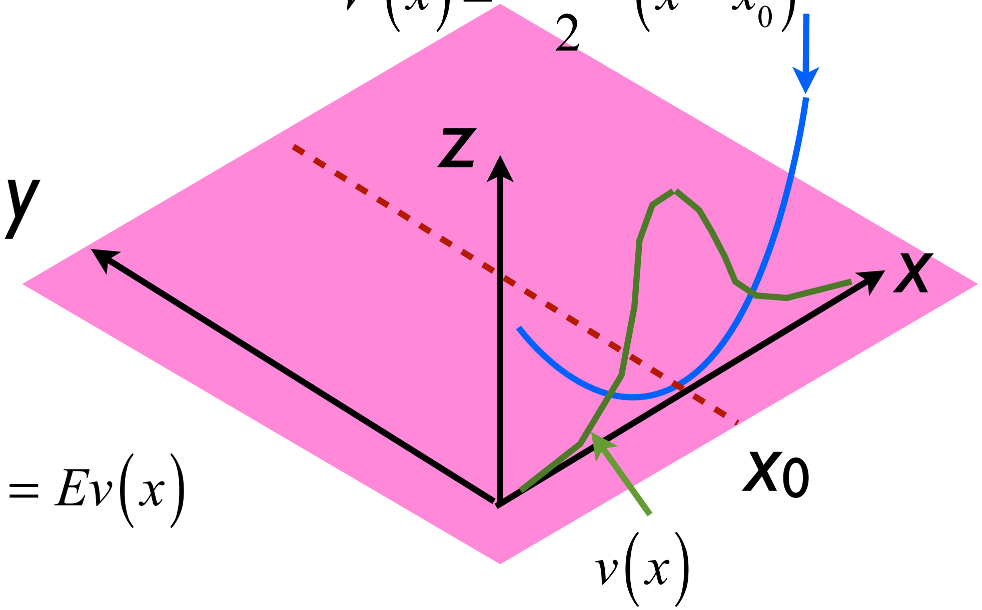
- Schrodinger eq

$$-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2}v(x)+\frac{e^2B^2}{2m_e}(x-x_0)^2 v(x) = Ev(x)$$

$$-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2}v(x)+\frac{m_e\omega^2}{2}(x-x_0)^2 v(x) = Ev(x)$$

effective potential

$$V(x) = \frac{m_e\omega^2}{2}(x-x_0)^2$$



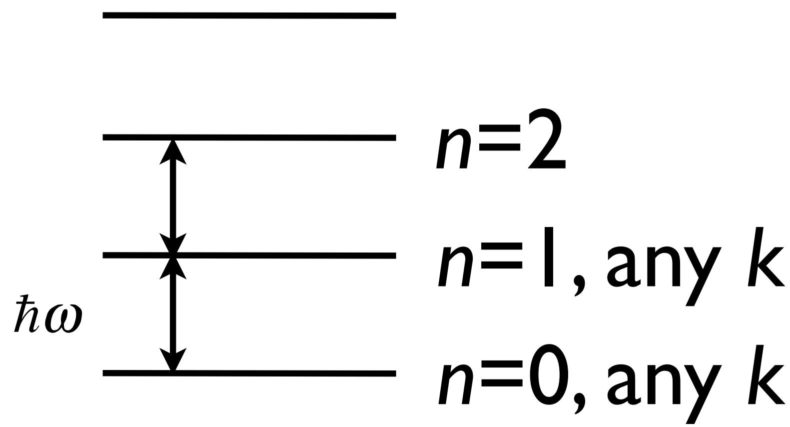
$$\omega = \frac{eB}{m_e}$$

Energy spectrum

- The corresponding eigenenergy

- Called Landau levels

$$E = \hbar\omega\left(n + \frac{1}{2}\right)$$

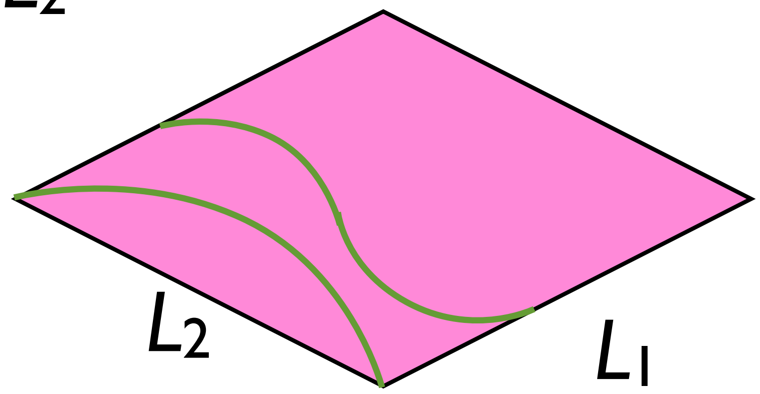


- The energy is independent on *k*! There will be a large number of degeneracy in each level

Confinement

- If the system has confinement in x and y directions. Lengths= L_1 and L_2
- We may take the boundary condition for y

$$\psi(x,y)=\psi(x,y+L_z)$$



- then k is quantized

$$\psi = e^{iky}v(x)$$

$$kL_2 = 2n^*\pi$$

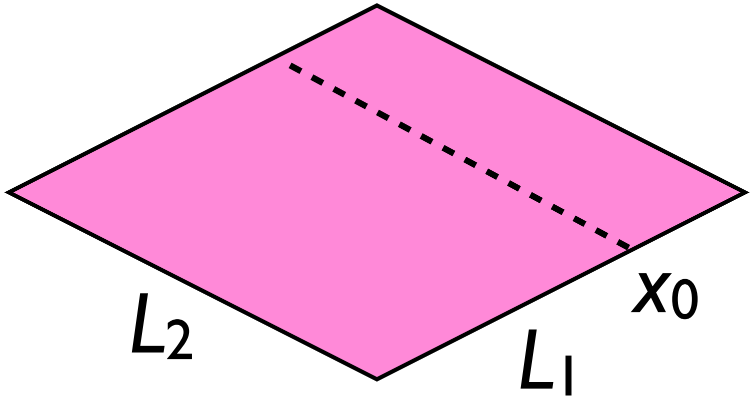
Confinement

- x_0 is quantized

$$x_0 = \frac{\hbar k}{eB} = \frac{2n^* \pi \hbar}{eBL_2}$$

- and x_0 is confined in L_1

$$0 < \frac{2n^* \pi \hbar}{eBL_2} < L_1$$



number of degeneracy

- The possible *n** are $$0 < n^* < \frac{eB}{2\pi\hbar} L_1 L_2$$

- Define the magnetic length as $$l_B = \sqrt{\frac{\hbar}{eB}}$$

meaning: the circular orbit radius $$\rho = \sqrt{\frac{2\hbar}{eB}} \quad \text{for } \ m = 1$$

- The maximal *n** value is the degeneracy number $$n^*_{\max} = \frac{L_1 L_2}{2\pi l_B^2}$$

Hall effect

- In a conductor, electric current follows Ohm's law

$$\mathbf{j} = \sigma_0 \mathbf{E}$$

- conductivity $\sigma_0 = \dfrac{n_e e^2 \tau_0}{m_e^*}$

τ_0 scattering time, or mean-free time

- In a magnetic field

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B} = \frac{1}{n_e} \mathbf{j} \times \mathbf{B}$$

$$\mathbf{j} = -en_e \mathbf{v}$$

- The current satisfies $\mathbf{j} = \sigma_0 \left(\mathbf{E} - \dfrac{1}{n_e e} \mathbf{j} \times \mathbf{B} \right)$

Setup for measurement

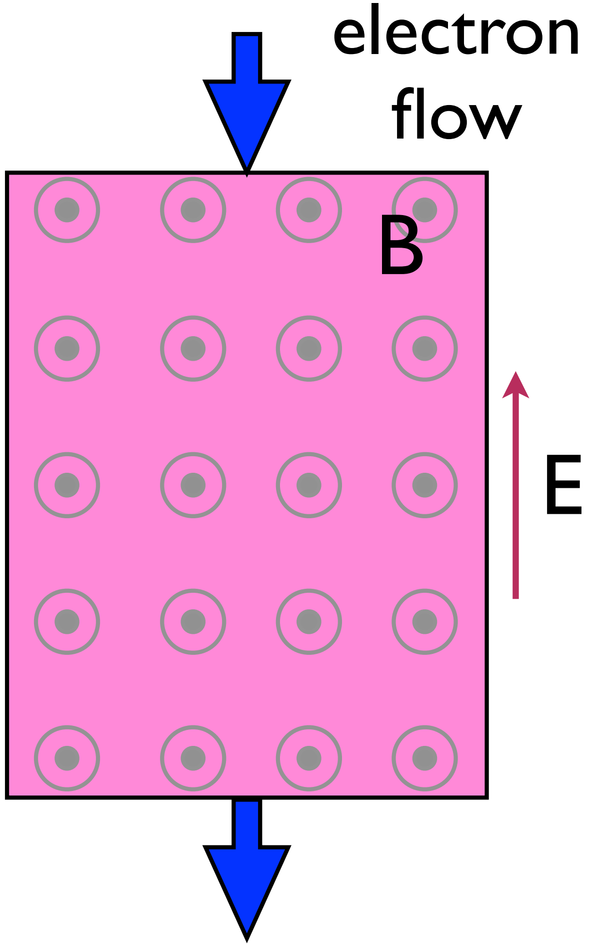
- *E* in *y*-direction, and *B* in the *z*-direction

$$j_x = -\frac{\sigma_0 j_y B}{n_e e}$$

$$j_y = \sigma_0 E_y + \frac{\sigma_0 j_x B}{n_e e} = \sigma_0 E_y - \left(\frac{\sigma_0 B}{n_e e}\right)^2 j_y$$

$$j_y = \frac{\sigma_0}{1+\left(\frac{\sigma_0 B}{n_e e}\right)^2} E_y = \frac{\sigma_0}{1+\left(\frac{e\tau_0 B}{m_e}\right)^2} E_y$$

$$j_x = \frac{\sigma_0 j_y B}{n_e e} = \frac{n_e e E_y}{B} \frac{\left(\frac{e\tau_0 B}{m_e}\right)^2}{1+\left(\frac{e\tau_0 B}{m_e}\right)^2}$$



Laudau level and electron density

- How many electrons per unit area?
- The maximum number of electron per unit area in a Landau level is

$$\frac{n^*_{\max}}{L_1 L} = \frac{1}{2\pi l_B^2} = \frac{eB}{2\pi\hbar}$$

- In any case, the electron density can be described by a filling factor

$$n_e = f\,\frac{eB}{2\pi\hbar}$$

transverse conductivity

- directional conductivity

$$\sigma_{yy} = \frac{j_y}{E_y} = \frac{\sigma_0}{1+\left(\frac{e\tau_0 B}{m_e}\right)^2}$$

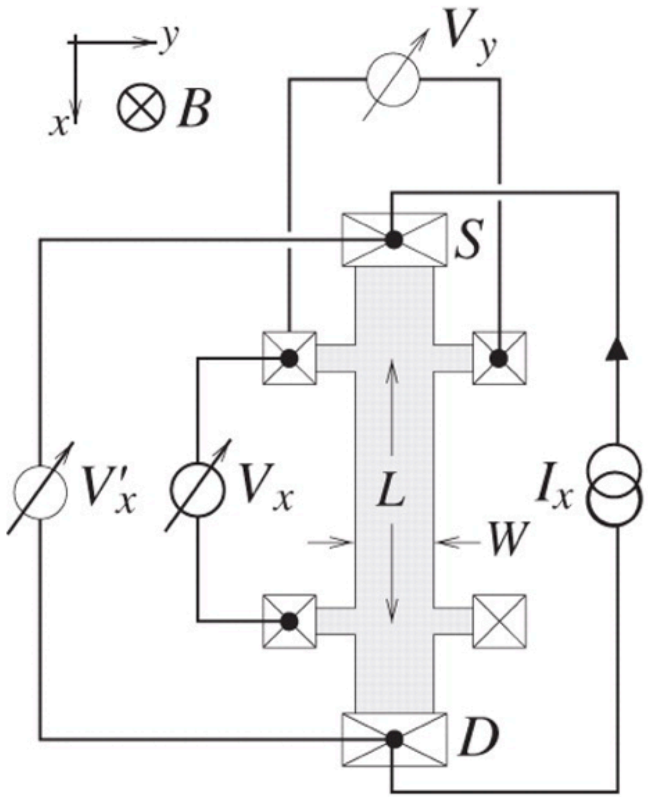
$$\sigma_{xy} = \frac{j_x}{E_y} = \frac{n_e e}{B}\frac{\left(\frac{e\tau_0 B}{m_e}\right)^2}{1-\left(\frac{e\tau_0 B}{m_e}\right)^2} = f\frac{e^2}{2\pi\hbar}\frac{\left(\frac{e\tau_0 B}{m_e}\right)^2}{1-\left(\frac{e\tau_0 B}{m_e}\right)^2}$$

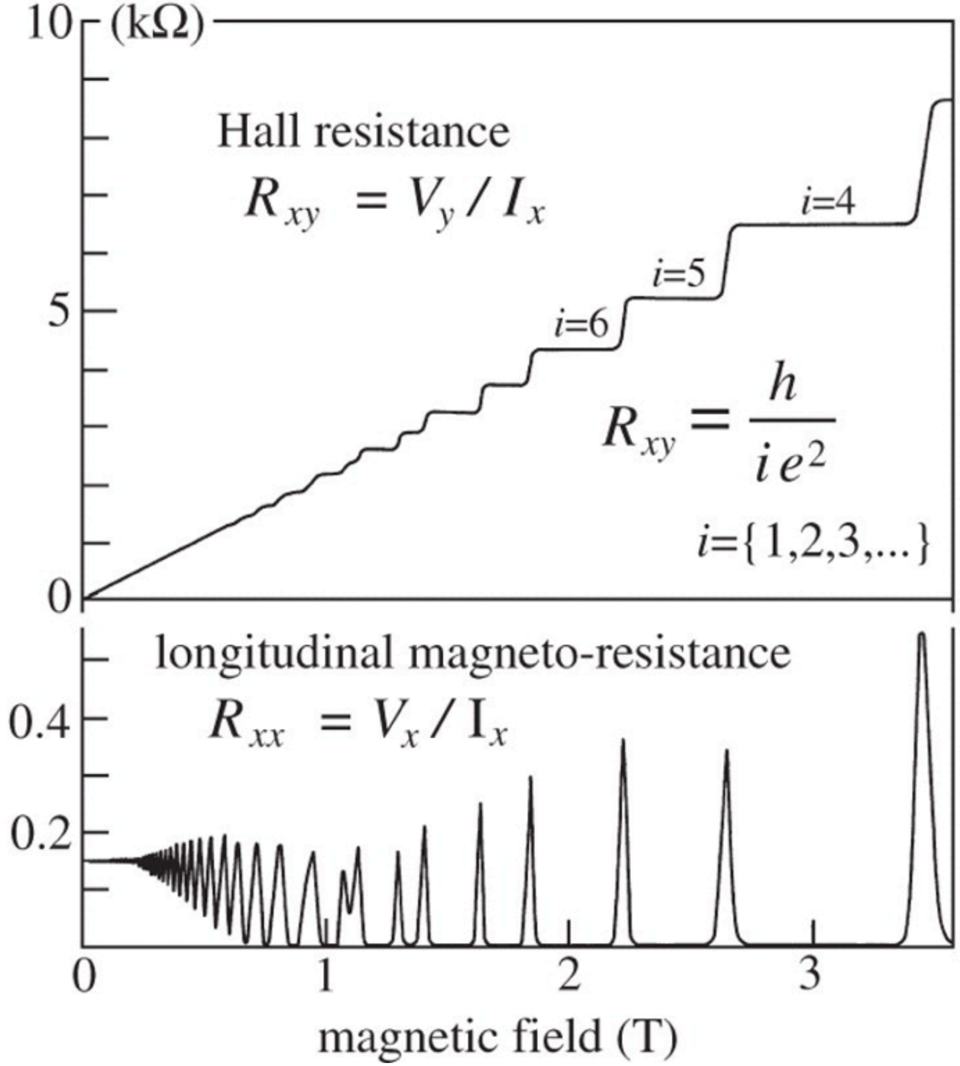
- when the Landau levels are fully filled, scattering time is infinite long

$$\sigma_{yy} \sim \infty$$

$$\sigma_{xy} = f\frac{e^2}{h} \qquad f = 1,2\cdots$$

Quantum Hall effect





$$R_{xy} = \frac{R_Q}{f} \qquad R_Q = \frac{h}{e^2} = 25.8 \text{ k}\Omega$$

confined magnetic field

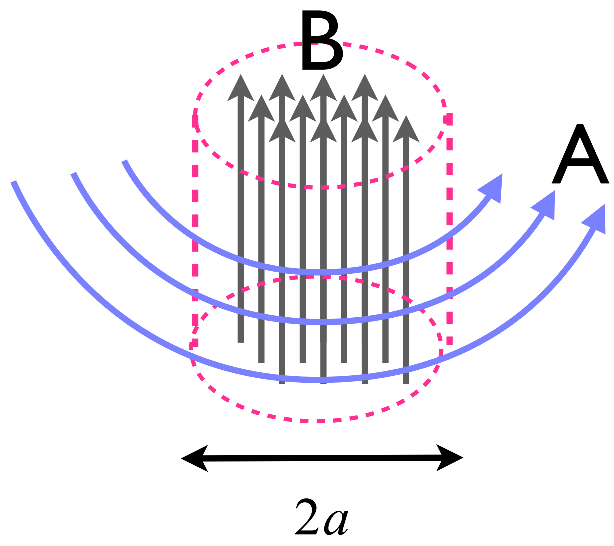
- Consider a constant magnetic field confined in a cylinder $\mathbf{B} = B\hat{z}$

- The vector potential takes the form when $\rho < a$

$$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B} = \frac{1}{2}B\rho\hat{\phi}$$

- When $\rho > a$ A is not zero!

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int \mathbf{B}\cdot d\mathbf{S} \qquad A_\phi = \frac{B\pi a^2}{2\pi\rho}$$



gauge transformation

- We may take a gauge transformation such that

$$\mathbf{A}' = \mathbf{A} - \nabla g = 0$$

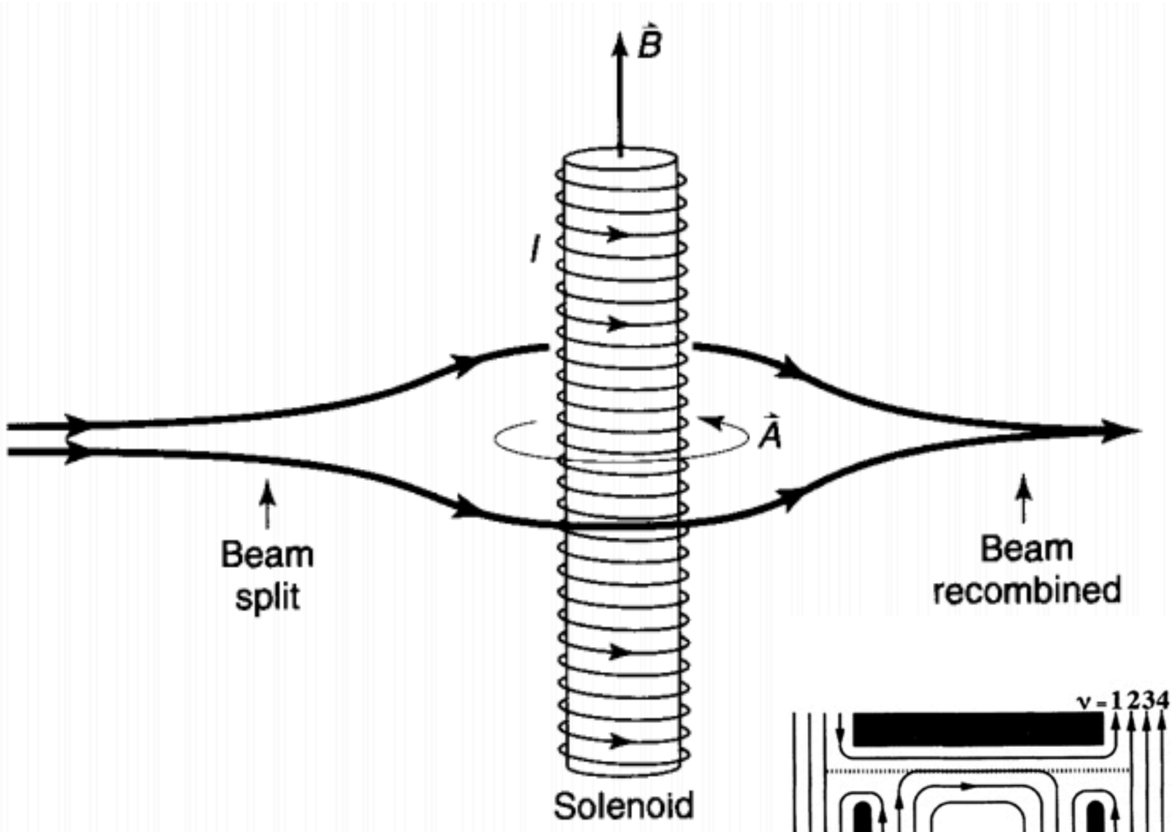
$$A_\phi = \frac{\Phi}{2\pi\rho} \qquad g = \frac{\Phi}{2\pi}\phi \qquad \Phi$$ magnetic flux

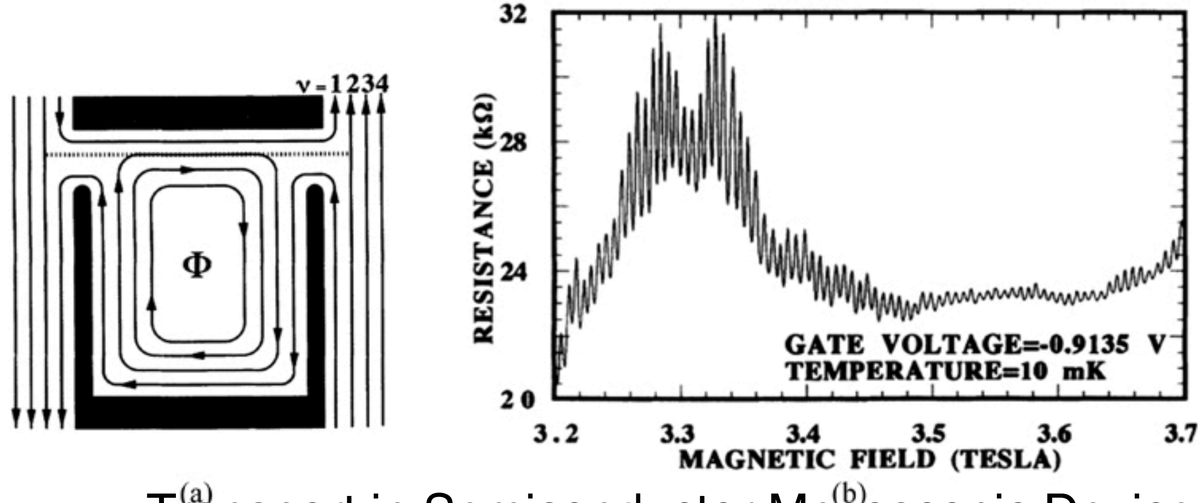
- With the gauge transformation, the wavefunction is transformed to

$$\psi' = e^{i\frac{e}{\hbar}g}\psi = e^{i\frac{e\Phi}{h}\phi}\psi$$

- The probability is unchanged, but could affect interference

Aharanov-Bohm effect





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