Electromagnetic waves

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Outlines

Waves in one dimension

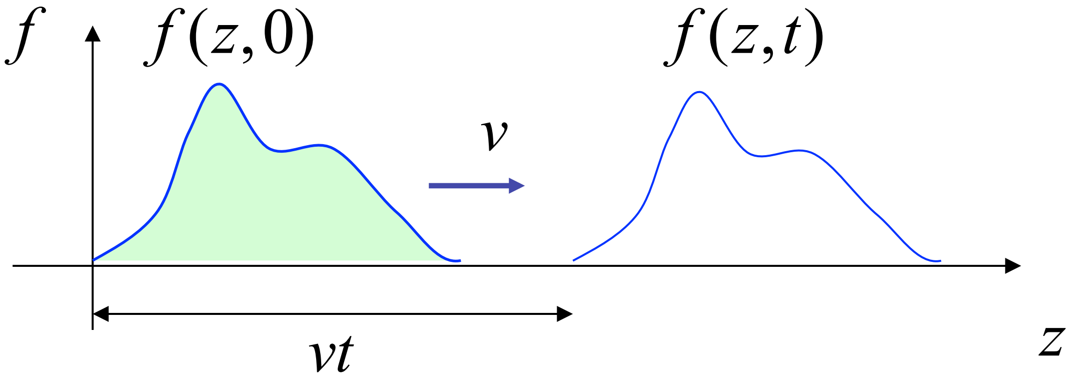
EM waves in vacuum

EM waves in matter

Absorption and dispersion

Propagation of a disturbance

Waves A disturbance of a continuous medium that propagates with a fixed shape at constant velocity



Initial condition $f(z,0)=g(z)$

What we call "disturbance shape"

Wave function

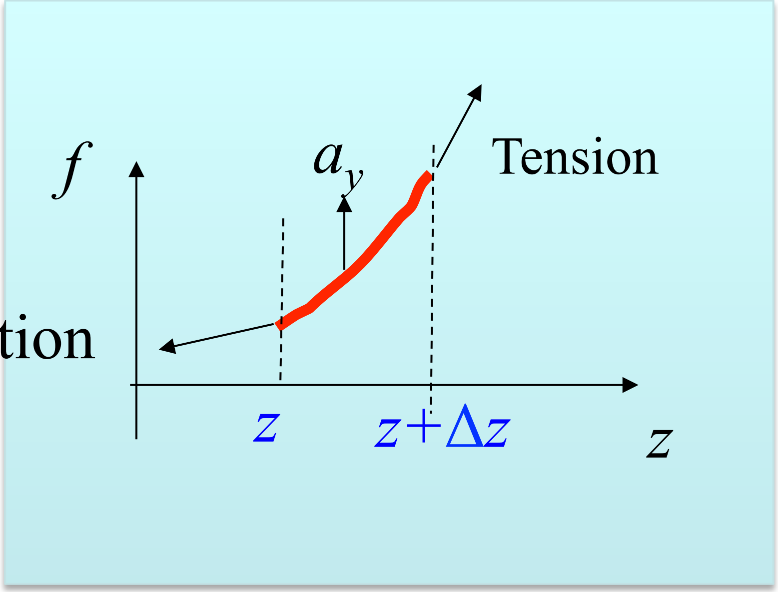
for any t $f(z,t)=f(z-vt,0)=g(z-vt)$

Wave equation

Here we consider the mechanics of a stretched string

The net force on a string segment

$$F = T\left(\frac{\partial f}{\partial z}\right)_{z+\Delta z} - T\left(\frac{\partial f}{\partial z}\right)_{z} = T\frac{\partial^2 f}{\partial z^2}\Delta z$$



The force produce an transverse acceleration

$$F = \mu\Delta z\frac{\partial^2 z}{\partial t^2}$$

$$\Rightarrow\quad T\frac{\partial^2 f}{\partial z^2}\Delta z = \mu\Delta z\frac{\partial^2 f}{\partial t^2}$$

or

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

$$v = \sqrt{\frac{T}{\mu}}$$

The above equation we call linear wave equation

Solutions to linear wave equation

Here we want to show that $f(z,t) = f(z - vt, 0) = g(z - vt)$
are the solutions to the wave equation

By using $u = z - vt$

$$\frac{\partial f\left(z,t\right)}{\partial z} = \frac{dg(u)}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}$$

$$\Rightarrow \qquad \frac{\partial^2 f\left(z,t\right)}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

Solutions to linear wave equation

$$\frac{\partial f\left(z,t\right)}{\partial t}=\frac{dg(u)}{du}\frac{\partial u}{\partial t}=-v\frac{dg}{du}$$

⇨

$$\frac{\partial^2 f\left(z,t\right)}{\partial t^2}=-v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t}=v^2\frac{d^2 g}{du^2}$$

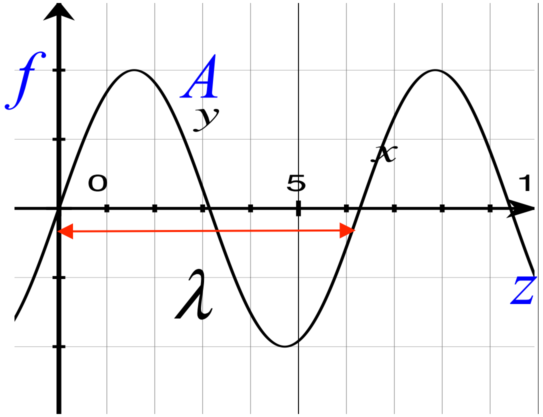
$$\frac{d^2 g}{du^2}=\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

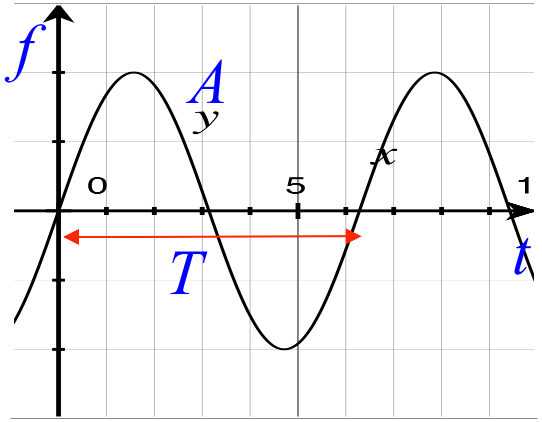
This is the linear
wave equation

One can also easily show that $f(z,t)=h(z+vt)$ for any shape $h(z)$ are also the solutions to the same wave equation

Meaning: propagation in the negative direction

Sinusoidal waves





Sinusoidal wave

$$f(z,t) = A\cos\left[k(z - vt) + \delta\right]$$

Wave length $\lambda = \dfrac{2\pi}{k}$

frequency ν $T = \dfrac{1}{\nu}$

Wave speed $v = \lambda\nu$

Amplitude A

Phase δ

$$f(z,t) = A\cos(kz - \omega t + \delta)$$ **Angular frequency** $\omega = kv$

Complex notation

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow \quad f(z,t) = \mathrm{Re}\left[A e^{i(kz-\omega t+\delta)} \right]$$

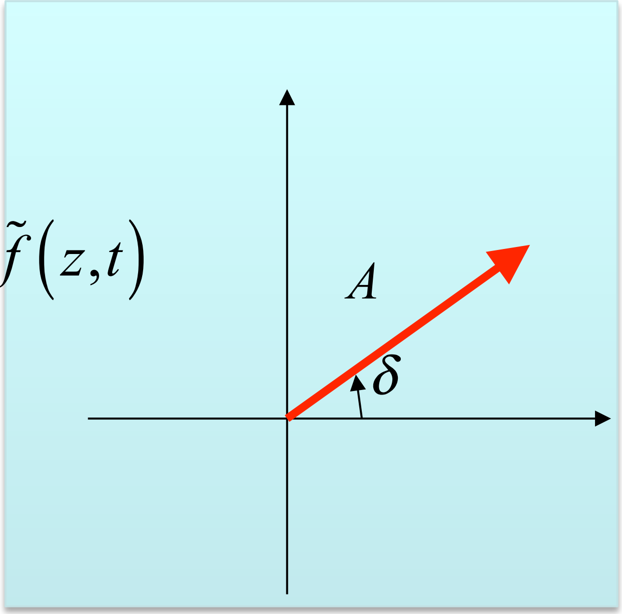
Introduce the complex wave function

$$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)}$$

with the complex amplitude $\tilde{A} = A e^{i\delta}$

The actual wave function is the real part of $\tilde{f}(z,t)$

$$f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t) \right]$$



Complex wave function algebra

The combination of two wave functions

$$f_3 = f_1 + f_2$$

When the complex wave functions are used, one has

$$f_3 = f_1 + f_2 = \mathrm{Re}\left(\tilde{f}_1\right) + \mathrm{Re}\left(\tilde{f}_2\right) = \mathrm{Re}\left(\tilde{f}_1 + \tilde{f}_2\right)$$

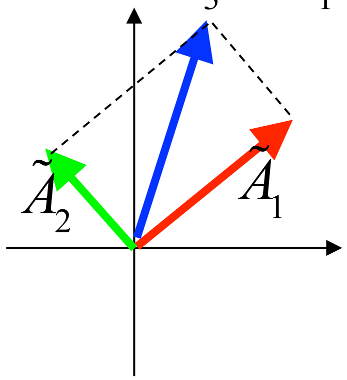
If define $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$ ⇒ $f_3 = \mathrm{Re}\left(\tilde{f}_3\right)$

In the case that three wave functions have the same k and ω

⇒ $$\tilde{A}_3 e^{i(kz-\omega t)} = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)}$$

And it can be simplified to $\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2$

$$\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2$$



Linear combinations of waves

In general, any wave can be expressed as a linear combination of sinusoidal ones

$$\tilde{f}\left(z,t\right)=\sum_{k}\tilde{A}\left(k\right)e^{i\left(kz-\omega t\right)}$$

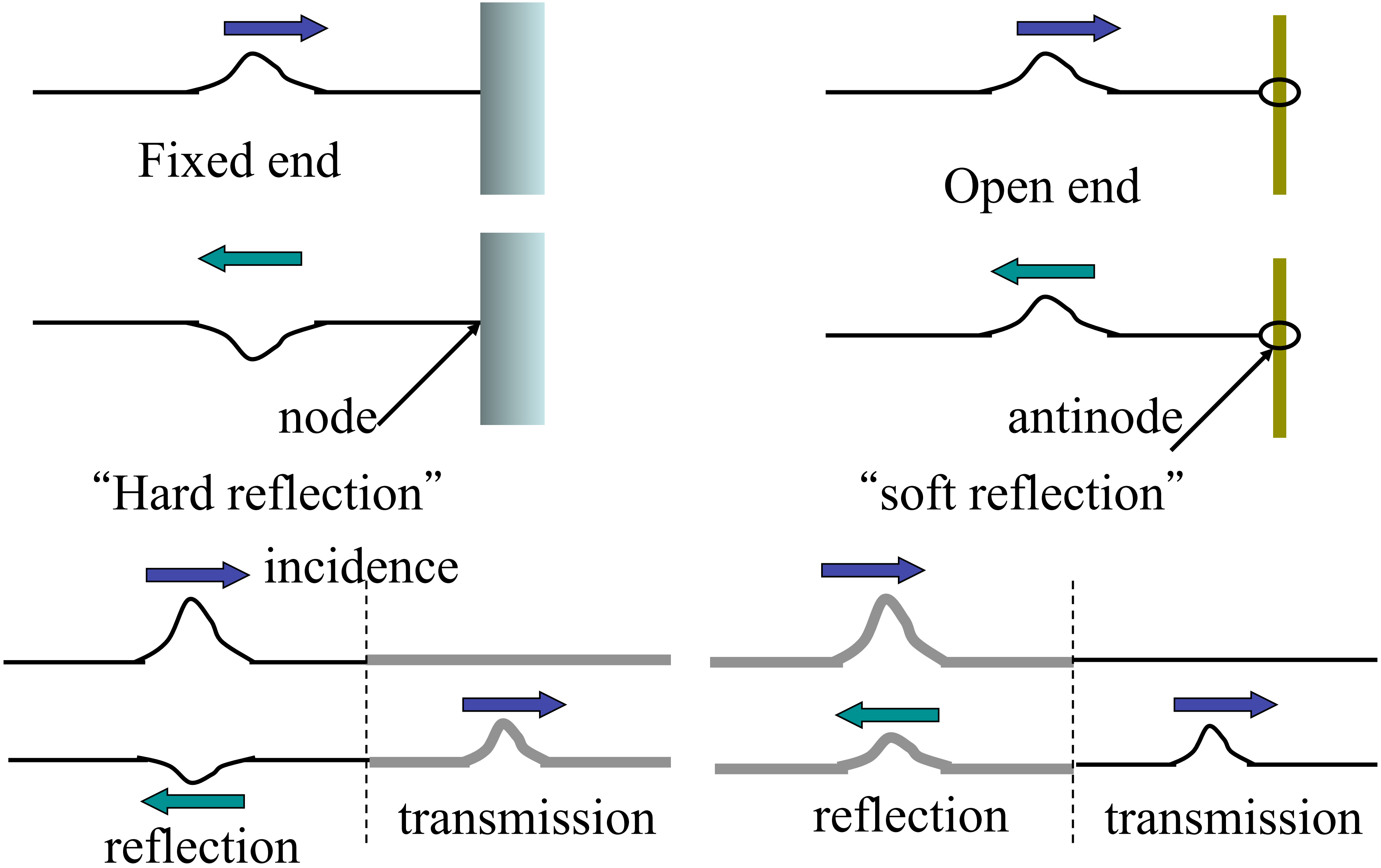
Since the wave number *k* is continuous, the above equation is better expressed by using an integral on *k*

⇒

$$\tilde{f}\left(z,t\right)=\int_{-\infty}^{\infty}\tilde{A}\left(k\right)e^{i\left(kz-\omega t\right)}dk$$

The completeness and uniqueness of the above expression need further proof. One can refer to Fourier's theorem

reflection and transmission of waves



Why?

Boundary conditions for strings

Now we impose two boundary conditions for *f* at z=0

$$f\left(0^+,t\right)=f\left(0^-,t\right)$$

$$\left.\frac{\partial f}{\partial z}\right|_{z=0^+}=\left.\frac{\partial f}{\partial z}\right|_{z=0^-}$$

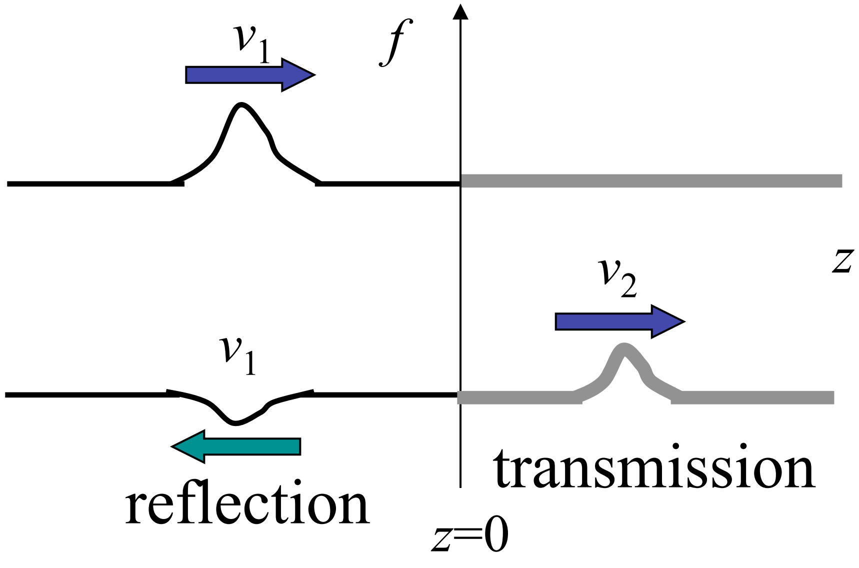
The conditions can be expressed in terms of $\tilde{f}$

$$\tilde{f}\left(0^+,t\right)=\tilde{f}\left(0^-,t\right) \qquad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{z=0^+}=\left.\frac{\partial \tilde{f}}{\partial z}\right|_{z=0^-}$$

$$\Rightarrow \quad \begin{aligned} &\tilde{A}_I+\tilde{A}_R=\tilde{A}_T \\ &k_1\left(\tilde{A}_I-\tilde{A}_R\right)=k_2\tilde{A}_T \end{aligned}$$

$$\Rightarrow \quad \tilde{A}_R=\frac{k_1-k_2}{k_1+k_2}\tilde{A}_I \qquad \tilde{A}_T=\frac{2k_1}{k_1+k_2}\tilde{A}_I$$

Boundary conditions for strings



The incident wave

$$\tilde{f}_I\left(z,t\right)=\tilde{A}_I e^{i\left(k_1 z-\omega t\right)} \qquad z<0$$

The reflected wave

$$\tilde{f}_R\left(z,t\right)=\tilde{A}_R e^{i\left(-k_1 z-\omega t\right)} \qquad z<0$$

The transmitted wave

$$\tilde{f}_T\left(z,t\right)=\tilde{A}_T e^{i\left(k_2 z-\omega t\right)} \qquad z>0$$

Notice that all parts of the system are oscillating with the same frequency w

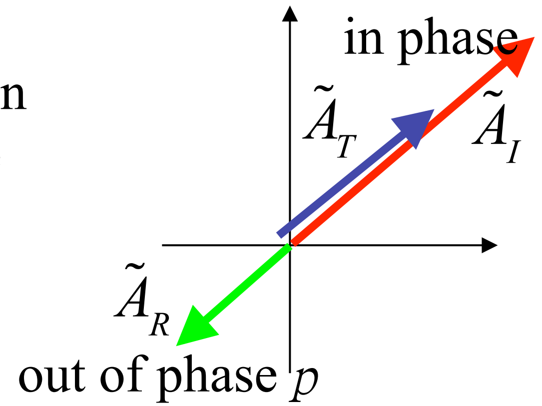
Then the total wave function on the string is

$$\tilde{f}\left(z,t\right)=\begin{cases} \tilde{A}_I e^{i\left(k_1 z-\omega t\right)}+\tilde{A}_R e^{i\left(-k_1 z-\omega t\right)} & \text{for } z<0 \\ \tilde{A}_T e^{i\left(k_2 z-\omega t\right)} & \text{for } z>0 \end{cases}$$

Reflection and transmission on strings

$$\tilde{A}_T = \frac{2v_2}{v_1 + v_2}\tilde{A}_I$$

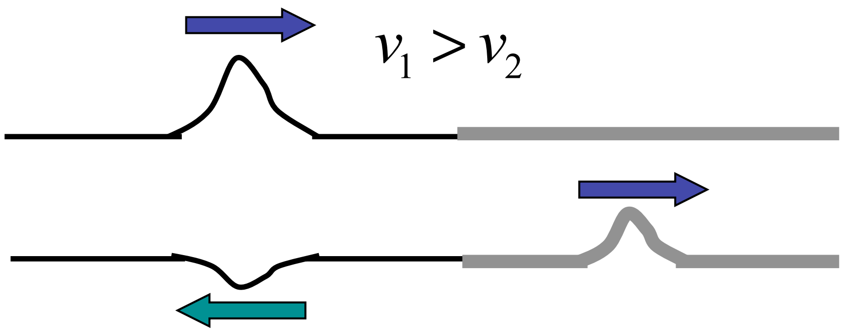
The transmitted wave is in phase with incident wave

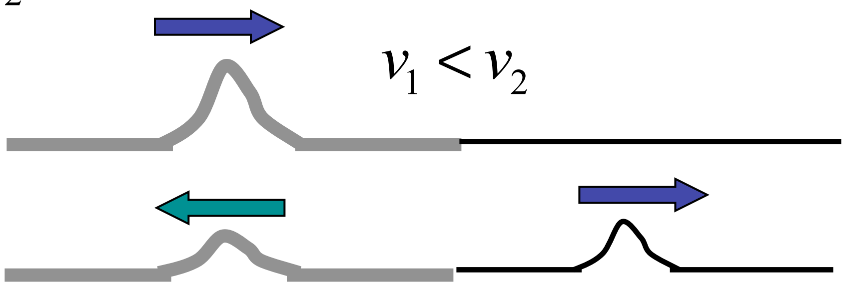


$$\tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2}\tilde{A}_I = \frac{v_2 - v_1}{v_1 + v_2}\tilde{A}_I$$

$$v_1 = \frac{\omega}{k_1} \qquad v_2 = \frac{k}{\omega_2}$$

The reflected wave is in phase with incident wave when $v_1 < v_2$ whereas is out of phase p when $v_1 > v_2$



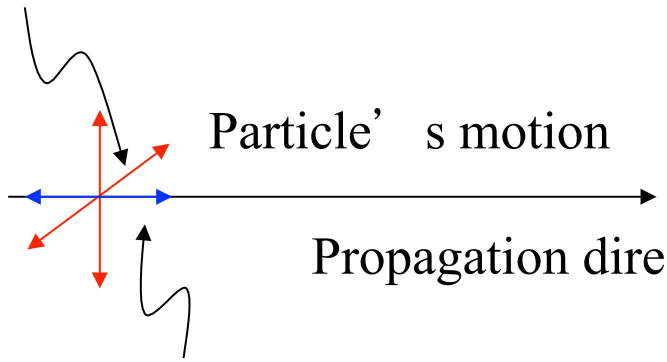


For fixed end, v_2=0 (or $\tilde{f}\left(z=0\right)=0$), so $\tilde{A}_R = -\tilde{A}_I$ $\tilde{A}_T$=0

For open end, v_2=infinite (or $\left.\frac{\partial \tilde{f}}{\partial z}\right|_{z=0} = 0$), so $\tilde{A}_R = \tilde{A}_I$ $\tilde{A}_T$=0

Wave polarization

Waves in three dimension



The red ones are called transverse waves

Particle’s motion

Propagation direction, *z*

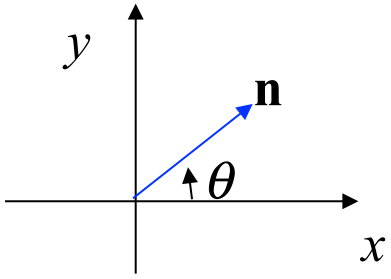
The blue one is called longitudinal wave

The polarization vector **n** defines the direction of vibration, $\mathbf{n}\cdot\mathbf{z}=0$

The wave function can be written as $\tilde{\mathbf{f}}\left(z,t\right)=\tilde{A}e^{i\left(k_2 z-\omega t\right)}\mathbf{n}$

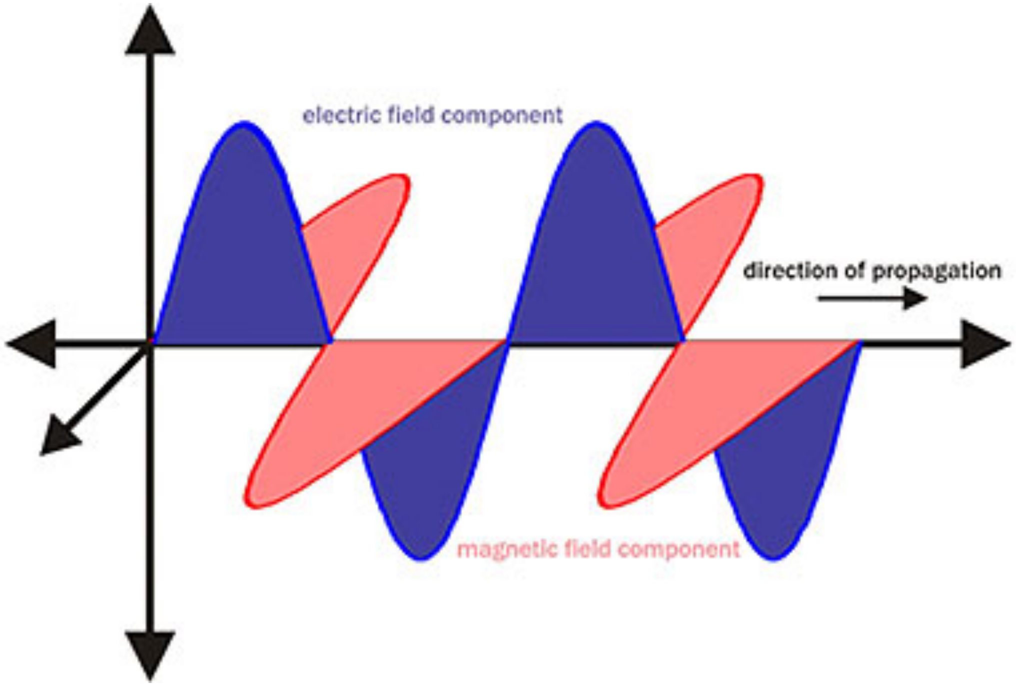
In general, the wave with **n** can be expressed by two fundamental polarizations

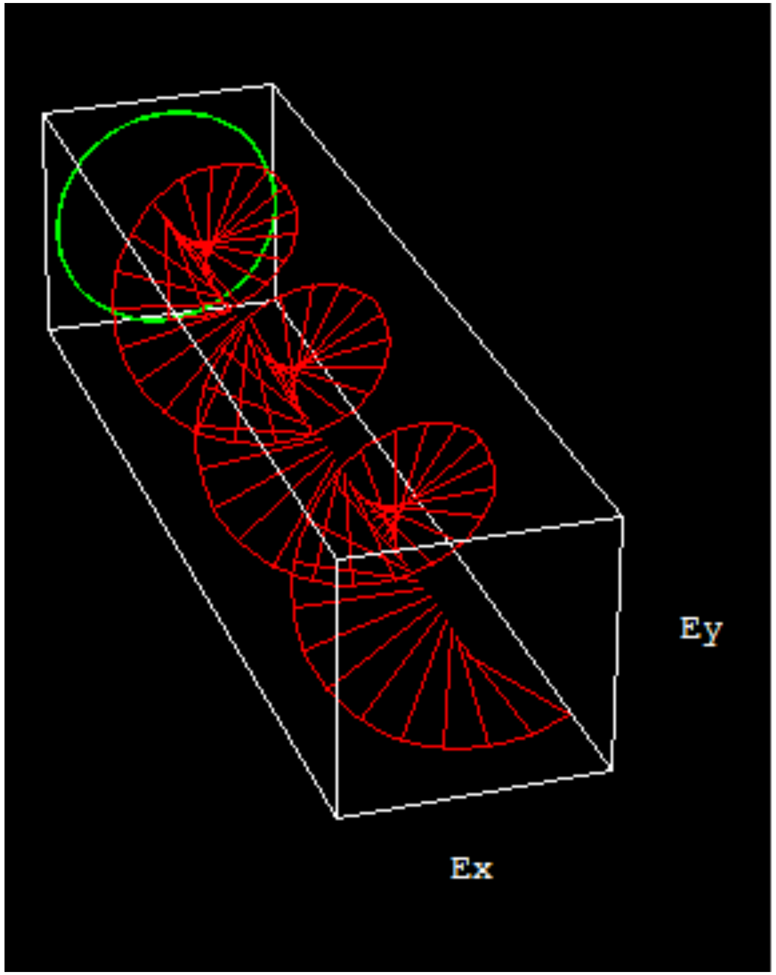
$$\tilde{\mathbf{f}}\left(z,t\right)=\tilde{A}\cos\theta e^{i\left(k_2 z-\omega t\right)}\mathbf{x}+\tilde{A}\sin\theta e^{i\left(k_2 z-\omega t\right)}\mathbf{y}$$



Circular polarization

Linear polarization





http://www.optics.arizona.edu/jcwyant/JoseDiaz/Polarization-Circular.htm

Waves in one dimension

EM waves in vacuum

EM waves in matter

Absorption and dispersion

Maxwell's equations in vacuum

The Maxwell's equations without charge and current

$$\nabla\cdot\mathbf{E}=0 \tag{1}$$

$$\nabla\cdot\mathbf{B}=0 \tag{2}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

If one applies the curl to (3)

$$\nabla\times\left(\nabla\times\mathbf{E}\right)=\nabla\left(\nabla\cdot\mathbf{E}\right)-\nabla^2\mathbf{E}=\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)$$

$$=-\frac{\partial}{\partial t}\left(\nabla\times\mathbf{B}\right)=-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla\cdot\mathbf{E}=0 \qquad \Longrightarrow \qquad \nabla^2\mathbf{E}=\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Wave equations for **E** and **B**

If one applies the curl to (4)

$$\nabla\times(\nabla\times\mathbf{B})=\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=\nabla\times\left(\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$=\mu_0\varepsilon_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})=-\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$\nabla\cdot\mathbf{B}=0$ ⇨ $\nabla^2\mathbf{B}=\mu_0\varepsilon_0\dfrac{\partial^2\mathbf{B}}{\partial t^2}$

E and **B** satisfy the 3D wave equation $\nabla^2 f=\dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$

⇨ The fields form waves and propagate in a speed of

$$c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=3.00\times10^8\ \mathrm{m/s}$$

Monochromatic plane waves

Monochromatic: sinusoidal wave function

Plane wave: traveling in *z* direction without *x* and *y* dependences

$$\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t)=\tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

There are constrains from Maxwell' s equations

$$\nabla\cdot\mathbf{E}=0 \quad \Rightarrow \quad \nabla\cdot\tilde{\mathbf{E}}(z,t)=0$$

$$\nabla\cdot\tilde{\mathbf{E}}(z,t)=\frac{\partial\left[\tilde{E}_{0,x}e^{i(kz-\omega t)}\right]}{\partial x}+\frac{\partial\left[\tilde{E}_{0,y}e^{i(kz-\omega t)}\right]}{\partial y}+\frac{\partial\left[\tilde{E}_{0,z}e^{i(kz-\omega t)}\right]}{\partial z}$$
$$=0+0+ik\tilde{E}_{0,z}$$

$$\Rightarrow \quad \tilde{E}_{0,z}=0$$

Similarly, $\tilde{B}_{0,z}=0$

Only $\frac{\partial\mathbf{E}}{\partial z}$ is non-zero

It follows that EM waves are transverse

The polarizations of **E** and **B**

From $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$

⇒ $$\nabla\times\tilde{\mathbf{E}} = -ik\tilde{E}_{0,y}e^{i(kz-\omega t)}\mathbf{x} + ik\tilde{E}_{0,x}e^{i(kz-\omega t)}\mathbf{y}$$

$$= i\omega\tilde{B}_{0,x}e^{i(kz-\omega t)}\mathbf{x} + i\omega\tilde{B}_{0,y}e^{i(kz-\omega t)}\mathbf{y}$$

Only $\frac{\partial\mathbf{E}}{\partial z}$ is non-zero

⇒ $$-k\tilde{E}_{0,y} = \omega\tilde{B}_{0,x}$$

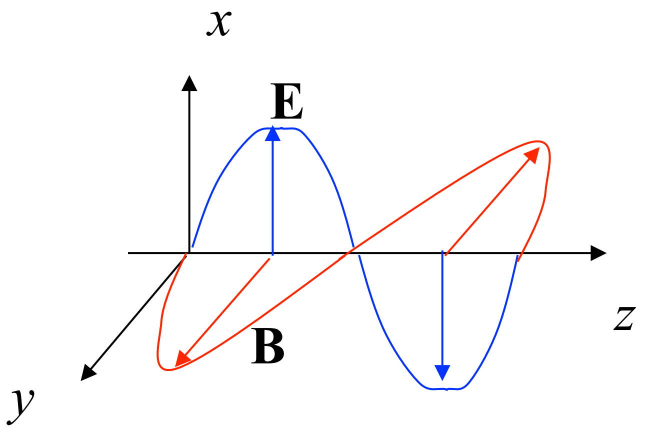
$$k\tilde{E}_{0,x} = \omega\tilde{B}_{0,y}$$

These can be expressed in a compact form

$$\boxed{\tilde{\mathbf{B}}_0 = \frac{k}{\omega}\left(\mathbf{z}\times\tilde{\mathbf{E}}_0\right)}$$

The amplitudes of **E** and **B** fields satisfy

$$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$$



Generalized plane wave functions

Introduce the propagation vector, **k**

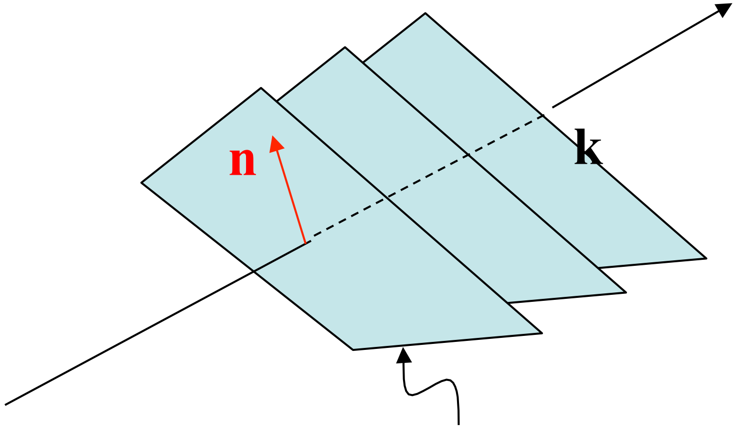
$$kz \to \mathbf{k}\cdot\mathbf{r}$$

$$\mathbf{z} \to \hat{\mathbf{k}}$$

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{n}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \tilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \frac{1}{c}E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left(\hat{\mathbf{k}}\times\mathbf{n}\right) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}$$

$$\mathbf{n}\cdot\mathbf{k} = 0$$



Wave front plane

Poynting’ s Theorem (I)

The energy stored in **E** and **B** fields is

$$U=\frac{1}{2}\int\left(\varepsilon_0 E^2+\frac{B^2}{\mu_0}\right)d\tau$$

Here we want to derive an energy conservation law for EM field. We start by considering the following conditions: If there exists a charge and current distribution, the EM field can do a work on the charges

$$dW=\mathbf{F}\cdot d\mathbf{l}=\left(q\mathbf{E}+q\mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v}dt=\mathbf{E}\cdot\left(\rho\mathbf{v}\right)d\tau dt$$

$$\Rightarrow \qquad \frac{dW}{dt}=\int\left(\mathbf{E}\cdot\mathbf{J}\right)d\tau$$

By using Ampere-Maxwell law, $\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$

$$\Rightarrow \qquad \mathbf{E}\cdot\mathbf{J}=\frac{1}{\mu_0}\mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right)-\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

Poynting’ s Theorem (II)

$$\mathbf{E}\cdot\mathbf{J}=-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})+\frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E})-\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

$$=-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})-\frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}-\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{B}) \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\mathbf{E}\cdot\mathbf{J}=-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})-\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)$$

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2}\frac{\partial B^2}{\partial t}$$

$$\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{2}\frac{\partial E^2}{\partial t}$$

$$\frac{dW}{dt}=\int(\mathbf{E}\cdot\mathbf{J})\,d\tau=-\frac{1}{\mu_0}\int_V\nabla\cdot(\mathbf{E}\times\mathbf{B})\,d\tau-\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)d\tau$$

$$=-\frac{1}{\mu_0}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}-\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)d\tau$$

Poynting’ s Theorem

Poynting’s Theorem (III)

The Poynting vector is defined by

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$ Unit: J/s m^2

It has the physical meaning of energy flux density at any point

$\mathbf{S}\cdot d\mathbf{a}$ is the energy flux through area element $d\mathbf{a}$ per unit time

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \int_S \mathbf{S}\cdot d\mathbf{a}$$

Work done on the charges

Energy stored in field

Energy flow

The work done on charge would increase the mechanical energy U_{mech}

$$\frac{dW}{dt} = \frac{dU_{mech}}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(U_{em} + U_{mech}\right) = -\int_S \mathbf{S}\cdot d\mathbf{a}$$

Energy conservation in electromagnetism

Energy in EM waves

The EM field energy density

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right)$$

For a monochromatic plane EM wave

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{E^2}{\mu_0 c^2} = \varepsilon_0 E^2$$

$$B = \frac{E}{c}$$

The energy flux density transported by the wave is

$$\mathbf{S} = \frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right)$$

$$\mathbf{S} = c\varepsilon_0 E^2 \mathbf{z} = cu\mathbf{z}$$

Light intensity

The energy density and energy flux is time oscillating, and one can find the time-averaged variables

$$\langle u \rangle = \varepsilon_0 \langle E^2 \rangle = \frac{1}{2}\varepsilon_0 E_0{}^2$$

$$\langle \mathbf{S} \rangle = c\varepsilon_0 \langle E^2 \rangle \mathbf{z} = \frac{1}{2} c\varepsilon_0 E_0{}^2 \mathbf{z}$$

The averaged power per unit area transported by a EM wave is called the intensity

$$I = \langle S \rangle = \frac{1}{2} c\varepsilon_0 E_0{}^2$$

Momentum in EM fields (I)

The total forces on charges

$$\mathbf{F} = \int (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau$$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

→ The force per unit volume is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \varepsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}$$

$$\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{f} = \varepsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \varepsilon_0 \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

$$= \varepsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

Momentum in EM fields (II)

$$\mathbf{f} = \boxed{\varepsilon_0 (\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\cdot\mathbf{B})\mathbf{B} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \varepsilon_0 \mathbf{E}\times(\nabla\times\mathbf{E})} - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

These terms can be written as a total divergence

$$\left(\nabla\cdot\overleftrightarrow{\mathbf{T}}\right)_j = \varepsilon_0\left[(\nabla\cdot\mathbf{E})E_j + (\mathbf{E}\cdot\nabla)E_j - \frac{1}{2}\nabla_j E^2\right]$$

T :Maxwell stress tensor

$$+\frac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})B_j + (\mathbf{B}\cdot\nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

$$\Longrightarrow \qquad \mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \varepsilon_0\mu_0 \frac{\partial}{\partial t}\mathbf{S}$$

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \varepsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau$$

From $\mathbf{F} + \dfrac{d\mathbf{P}}{dt} = 0$ $\Longrightarrow$ $\varepsilon_0\mu_0 \int_{\mathrm{V}} \mathbf{S}\,d\tau$ is the momentum stored in EM fields

Radiation pressure

In monochromatic plane EM waves

$$\mathbf{f} = \varepsilon_0 \left(\cancel{\nabla \cdot \mathbf{E}}\right)\mathbf{E} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \varepsilon_0\mathbf{E}\times\left(\nabla\times\mathbf{E}\right) - \varepsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right)$$

$$= \frac{1}{\mu_0}\left(i\mathbf{k}\times\mathbf{B}\right)\times\mathbf{B} - \varepsilon_0\mathbf{E}\times\left(i\mathbf{k}\times\mathbf{E}\right) - \varepsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{S}$$

$$= -2i\omega\varepsilon_0\mathbf{E}\times\mathbf{B} - \varepsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{S}$$

$$\mathbf{F} + \frac{d\mathbf{P}}{dt} = 0 \qquad\qquad \frac{\partial}{\partial t}\mathbf{S} = -i\omega\mathbf{S}$$

The momentum density is $\mathbf{p} = \varepsilon_0\mu_0\mathbf{S} = \frac{u}{c}\mathbf{z}$ $\qquad \langle\mathbf{p}\rangle = \frac{1}{2c}\varepsilon_0 E_0{}^2\mathbf{z}$

The radiation pressure is $P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{A}\frac{1}{\Delta t}\left(\frac{1}{2c}\varepsilon_0 E_0{}^2\right)Ac\Delta t = \frac{\varepsilon_0 E_0{}^2}{2} = \frac{I}{c}$

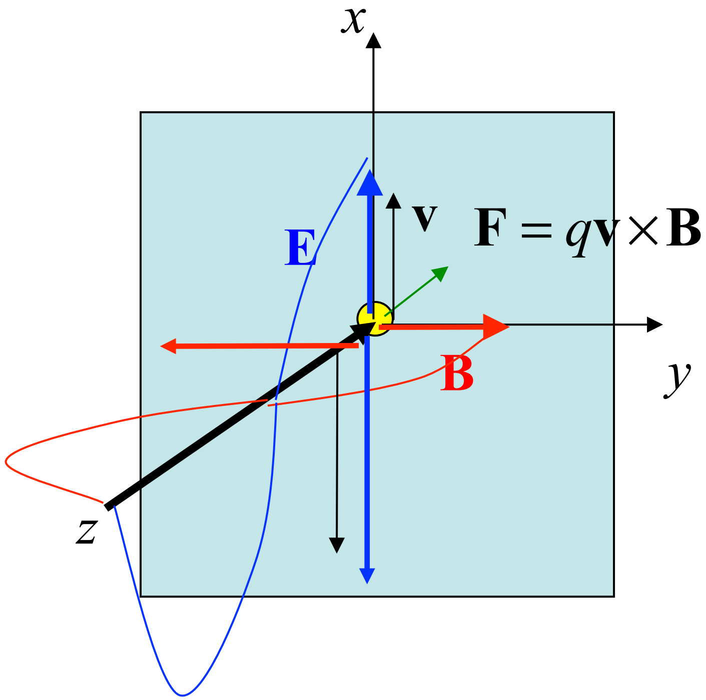
Physical interpretation of radiation pressure

The **E** field drives the charge in the *x* direction, producing a velocity of **v**

The **B** field further exerts a Lorentz force on the charge with in the *z* direction.

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$$

It can be shown that over a full cycle of oscillation, the in-plane force which is produced by **E** is averaged to zero, while the out-plane force is not.



Waves in one dimension

EM waves in vacuum

EM waves in matter

Absorption and dispersion

Propagation in linear and homogenous media

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu}$$

$$\Rightarrow$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow$$

The EM wave speed is replaced by

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

n: Index of refraction

Energy and momentum of EM waves in media

The energy density

$$u = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\frac{B^2}{\mu}$$

The energy flux density

$$\mathbf{S} = \frac{1}{\mu}\left(\mathbf{E}\times\mathbf{B}\right)$$

The intensity

$$I = \langle S \rangle = \frac{1}{2}\varepsilon v {E_0}^2$$

Boundary conditions for EM waves in media

No free charges and currents on the interfaces

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = 0$$

The reflection and transmission of EM waves at normal incidence

Assume a linear polarized EM wave

$$\tilde{\mathbf{E}}_I(z,t)=\tilde{E}_{0,I}e^{i(k_1z-\omega t)}\hat{\mathbf{x}}$$

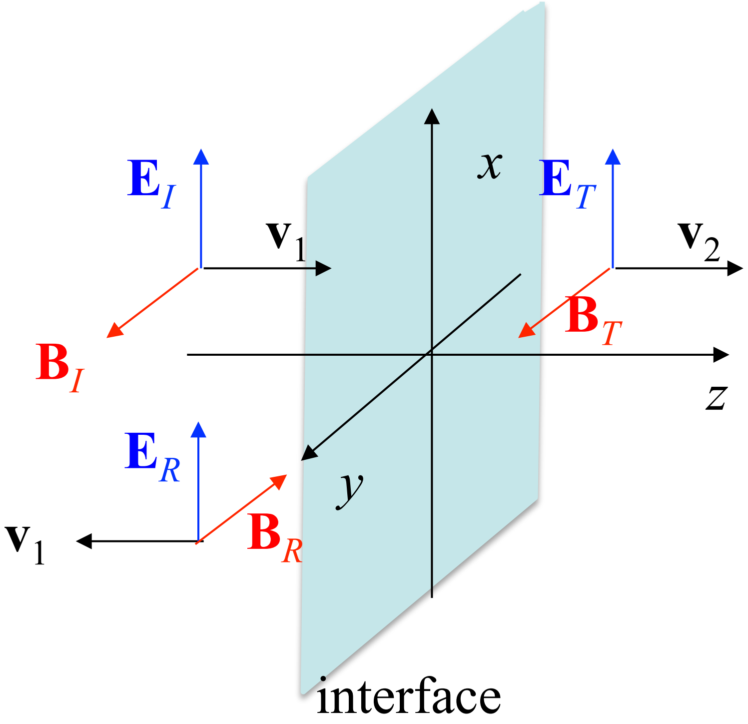
$$\tilde{\mathbf{B}}_I(z,t)=\frac{1}{v_1}\tilde{E}_{0,I}e^{i(k_1z-\omega t)}\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_T(z,t)=\tilde{E}_{0,T}e^{i(k_2z-\omega t)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_T(z,t)=\frac{1}{v_2}\tilde{E}_{0,T}e^{i(k_2z-\omega t)}\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_R(z,t)=\tilde{E}_{0,R}e^{i(-k_1z-\omega t)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_R(z,t)=-\frac{1}{v_1}\tilde{E}_{0,R}e^{i(-k_1z-\omega t)}\hat{\mathbf{y}}$$



The boundary conditions

At z=0

$$\tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}$$

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0,I} - \frac{1}{v_1}\tilde{E}_{0,R}\right) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0,T}$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = 0$$

$\Rightarrow$

$$\tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}$$

$$\tilde{E}_{0,I} - \tilde{E}_{0,R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0,T}$$

$$\tilde{E}_{0,R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} = \frac{2}{1+\beta}\tilde{E}_{0,I}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Analogy to string waves

For string waves

$$\tilde{A}_R = \frac{v_2 - v_1}{v_1 + v_2}\tilde{A}_I$$

$$\tilde{A}_T = \frac{2v_2}{v_1 + v_2}\tilde{A}_I$$

Normally incident EM waves

$$\tilde{E}_{0,R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} = \frac{2}{1+\beta}\tilde{E}_{0,I}$$

When $\mu_1 \simeq \mu_2$ $\quad \beta = \dfrac{v_1}{v_2}$

⇒

$$\tilde{E}_{0,R} = \frac{v_2 - v_1}{v_1 + v_2}\tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} = \frac{2v_2}{v_1 + v_2}\tilde{E}_{0,I}$$

In terms of refraction indices

$$\tilde{E}_{0,R} = \frac{n_1 - n_2}{n_1 + n_2}\tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} = \frac{2n_1}{n_1 + n_2}\tilde{E}_{0,I}$$

Reflection and transmission coefficients

$$I = \frac{1}{2}\varepsilon v E_0{}^2$$

⇒

$$R = \frac{I_R}{I_I} = \left|\frac{\tilde{E}_{0,R}}{\tilde{E}_{0,I}}\right|^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

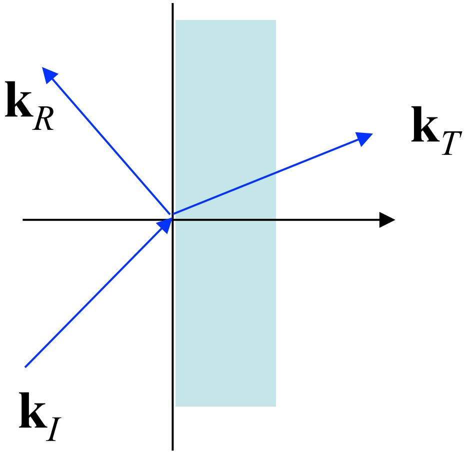
The reflection coefficient

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left|\frac{\tilde{E}_{0,T}}{\tilde{E}_{0,I}}\right|^2 = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4n_1 n_2}{\left(n_1 + n_2\right)^2}$$

The transmission coefficient

$$R + T = 1$$

The reflection and transmission at oblique incidence



For monochromatic plane waves

$$\tilde{\mathbf{E}}_I(\mathbf{r},t)=\tilde{\mathbf{E}}_{0,I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r},t)=\frac{1}{v_1}\left(\hat{\mathbf{k}}_I\times\tilde{\mathbf{E}}_I\right)$$

The reflected wave

$$\tilde{\mathbf{E}}_R(\mathbf{r},t)=\tilde{\mathbf{E}}_{0,R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_R(\mathbf{r},t)=\frac{1}{v_1}\left(\hat{\mathbf{k}}_R\times\tilde{\mathbf{E}}_R\right)$$

$$k_I v_1=k_R v_1=k_T v_2=\omega$$

The transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r},t)=\tilde{\mathbf{E}}_{0,T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_T(\mathbf{r},t)=\frac{1}{v_2}\left(\hat{\mathbf{k}}_T\times\tilde{\mathbf{E}}_T\right)$$

$$k_I=k_R=\frac{n_1}{n_2}k_T$$

The time-varying phase part

The time-dependent parts of the field should obey

$$\left(\cdots\right)e^{i\left(\mathbf{k}_I\cdot\mathbf{r}-\omega t\right)}+\left(\cdots\right)e^{i\left(\mathbf{k}_R\cdot\mathbf{r}-\omega t\right)}=\left(\cdots\right)e^{i\left(\mathbf{k}_T\cdot\mathbf{r}-\omega t\right)}$$ at z=0

⇒

$$e^{i\left(\mathbf{k}_I\cdot\mathbf{r}-\omega t\right)}=e^{i\left(\mathbf{k}_R\cdot\mathbf{r}-\omega t\right)}=e^{i\left(\mathbf{k}_T\cdot\mathbf{r}-\omega t\right)}$$

at z=0

$$\mathbf{k}_I\cdot\mathbf{r}=\mathbf{k}_R\cdot\mathbf{r}=\mathbf{k}_T\cdot\mathbf{r}$$

Orient the axes so that $\mathbf{k}_I$ lies in the x-z plane, one has

$$k_{I,y}=k_{R,y}=k_{T,y}=0 \qquad \mathbf{r}\parallel\hat{\mathbf{y}}$$

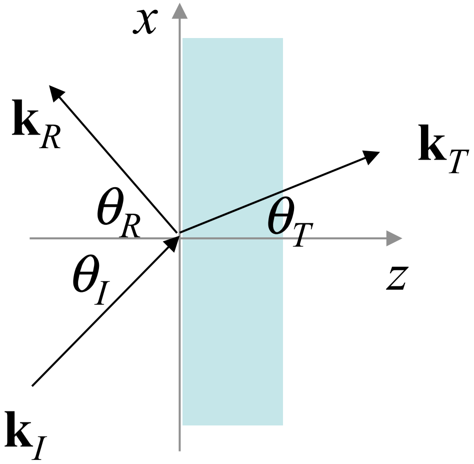
The incident, reflected, and transmitted wave vectors form a plane, which also include the normal to the surface (z-direction)

Laws of reflection and refraction

$$k_{I,x} = k_{R,x} = k_{T,x} \qquad \mathbf{r} \parallel \hat{\mathbf{x}}$$

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Phase matching condition



$$\Rightarrow \quad \theta_I = \theta_R$$ Law of reflection

$$\Rightarrow \quad n_1 \sin\theta_I = n_2 \sin\theta_T$$ Law of refraction

Boundary conditions for the time-independent part

The time-independent parts of the field should obey

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \qquad \varepsilon_1\left(\tilde{\mathbf{E}}_{0,I} + \tilde{\mathbf{E}}_{0,R}\right)_z = \varepsilon_2\left(\tilde{\mathbf{E}}_{0,T}\right)_z \qquad (1)$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \left(\tilde{\mathbf{B}}_{0,I} + \tilde{\mathbf{B}}_{0,R}\right)_z = \left(\tilde{\mathbf{B}}_{0,T}\right)_z \qquad (2)$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \qquad \left(\tilde{\mathbf{E}}_{0,I} + \tilde{\mathbf{E}}_{0,R}\right)_{x,y} = \left(\tilde{\mathbf{E}}_{0,T}\right)_{x,y} \qquad (3)$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = 0 \qquad \frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_{0,I} + \tilde{\mathbf{B}}_{0,R}\right)_{x,y} = \frac{1}{\mu_2}\left(\tilde{\mathbf{B}}_{0,T}\right)_{x,y} \qquad (4)$$

$$\tilde{\mathbf{B}}_0 = \frac{1}{v}\hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0$$

Polarization in parallel to the plane of incidence

Consider the polarization in parallel to the plane of incidence:

$$\left(\tilde{\mathbf{E}}_{0,I}\right)_y=\left(\tilde{\mathbf{E}}_{0,R}\right)_y=\left(\tilde{\mathbf{E}}_{0,T}\right)_y=0$$

$$\left(\tilde{\mathbf{B}}_{0,I}\right)_x=\left(\tilde{\mathbf{B}}_{0,R}\right)_x=\left(\tilde{\mathbf{B}}_{0,T}\right)_x=0$$

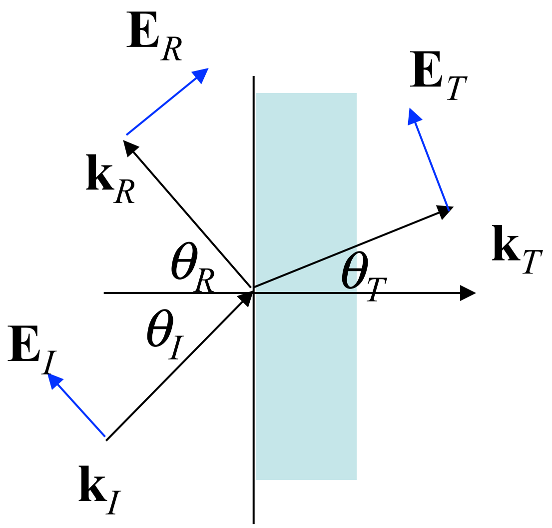
$$\Rightarrow \quad \left(\tilde{\mathbf{B}}_{0,I}\right)_z=\left(\tilde{\mathbf{B}}_{0,R}\right)_z=\left(\tilde{\mathbf{B}}_{0,T}\right)_z=0$$

Also called TM mode

(1) $$\varepsilon_1\left(-\tilde{E}_{0,I}\sin\theta_I+\tilde{E}_{0,R}\sin\theta_R\right)=-\varepsilon_2\tilde{E}_{0,T}\sin\theta_T$$

(3) $$\tilde{E}_{0,I}\cos\theta_I+\tilde{E}_{0,R}\cos\theta_R=\tilde{E}_{0,T}\cos\theta_T$$

(4) $$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0,I}-\tilde{E}_{0,R}\right)=\frac{1}{\mu_2 v_2}\tilde{E}_{0,T}$$



Fresnel equations

$$\varepsilon_1\left(-\tilde{E}_{0,I}\sin\theta_I+\tilde{E}_{0,R}\sin\theta_R\right)=-\varepsilon_2\tilde{E}_{0,T}\sin\theta_T$$

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0,I}-\tilde{E}_{0,R}\right)=\frac{1}{\mu_2 v_2}\tilde{E}_{0,T} \quad\Longrightarrow\quad \tilde{E}_{0,I}-\tilde{E}_{0,R}=\beta\tilde{E}_{0,T}$$

$$\tilde{E}_{0,I}\cos\theta_I+\tilde{E}_{0,R}\cos\theta_R=\tilde{E}_{0,T}\cos\theta_T \quad\Longrightarrow\quad \tilde{E}_{0,I}+\tilde{E}_{0,R}=\frac{\cos\theta_T}{\cos\theta_I}\tilde{E}_{0,T}$$

$$\tilde{E}_{0,R}=\frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0,I}$$

in phase or 180° out of phase with incident wave

$$\tilde{E}_{0,T}=\frac{2}{\alpha+\beta}\tilde{E}_{0,I}$$

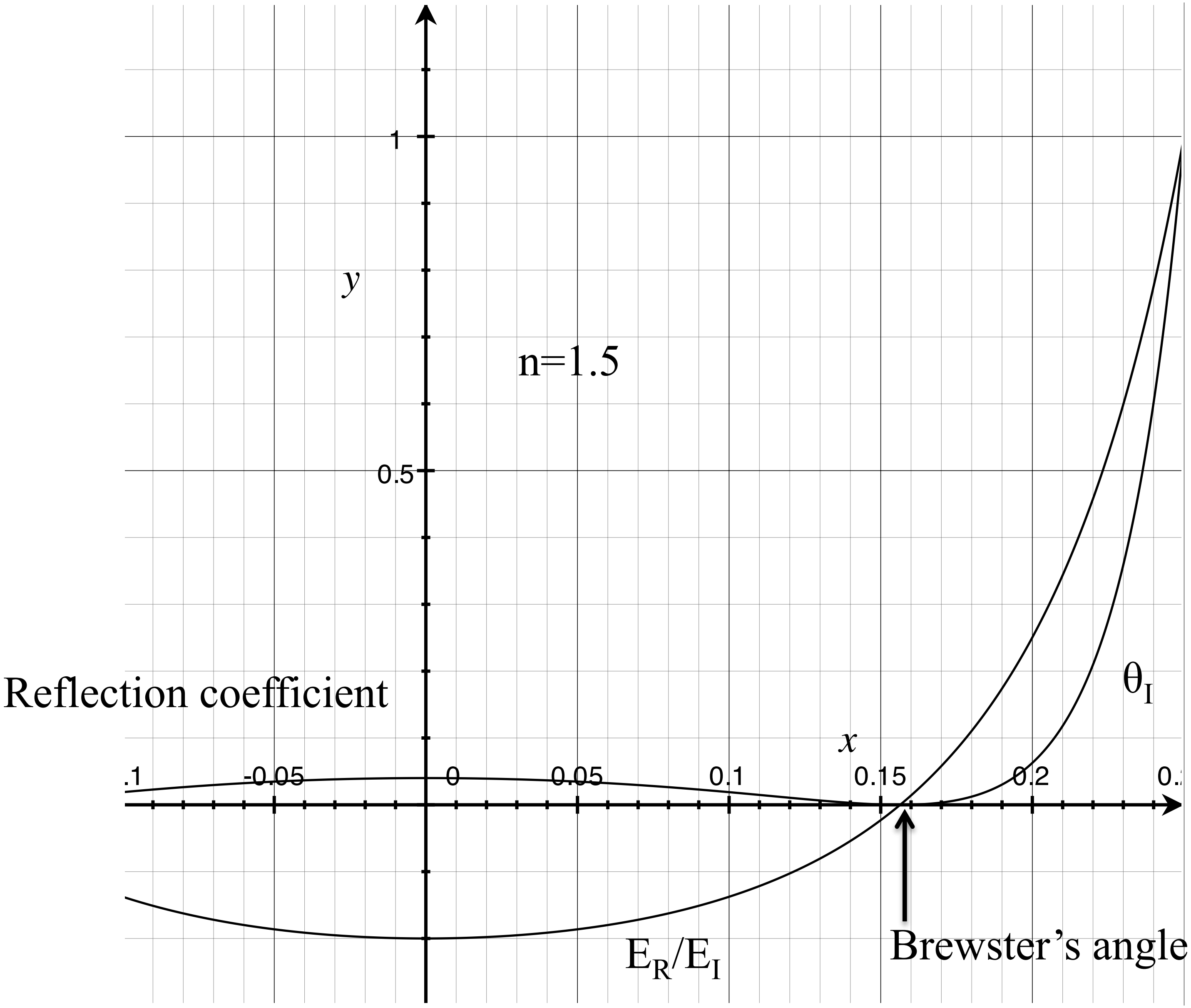
in phase with incident wave

$$\beta=\frac{\mu_1 v_1}{\mu_2 v_2}$$

$$\alpha=\frac{\cos\theta_T}{\cos\theta_I}$$

$$\alpha=\frac{\cos\theta_T}{\cos\theta_I}=\frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I}=\frac{\sqrt{1-\left(n_1/n_2\right)^2\sin^2\theta_I}}{\cos\theta_I}$$

One may derive the similar expressions for TE mode


Reflection coefficient
n=1.5
y
x
1
0.5
.1
-0.05
0
0.05
0.1
0.15
0.2
θ_I
E_R/E_I
Brewster's angle

Brewster’ s angle

In the case of normal incidence $\theta_I = 0$ $\alpha = 1$

In the case of grazing incidence $\theta_I = 90^{\circ}$ $\alpha = \infty$

$$\tilde{E}_{0,R} = \tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} = 0$$

⇒ Total reflection

The condition that reflected wave is extinguished $\alpha = \beta$

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

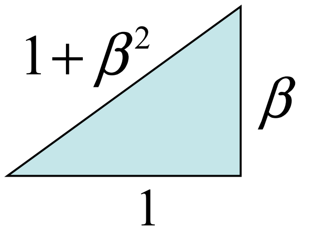
Brewster’ s angle

$$\mu_1 \simeq \mu_2$$

⇒ $\beta = \dfrac{n_2}{n_1}$

$$\sin^2\theta_B = \frac{1-\beta^2}{\beta^{-2} - \beta^2} = \frac{\beta}{1+\beta^2}$$

$$\tan\theta_B \simeq \frac{n_2}{n_1}$$



Reflection and transmission coefficients

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}{}^2 \cos\theta_I$$

$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}{}^2 \cos\theta_R$$

The reflection coefficient

$$I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}{}^2 \cos\theta_T$$

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0,R}}{E_{0,I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0,T}}{E_{0,I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}$$

$$R + T = 1$$

Waves in one dimension

EM waves in vacuum

EM waves in matter

Absorption and dispersion

EM waves in conductors

In general the free current is not zero

$$\mathbf{J}_f = \sigma \mathbf{E}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon} \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu \sigma \mathbf{E} + \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$$

The charge dissipation in conductors

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t} \quad \Longrightarrow \quad \frac{\partial \rho_f}{\partial t} = -\sigma\left(\nabla \cdot \mathbf{E}\right) = -\frac{\sigma}{\varepsilon}\rho_f$$

$$\rho_f\left(t\right) = \rho_f\left(0\right)e^{-(\sigma/\varepsilon)t}$$

Characteristic time scale $\tau = \dfrac{\varepsilon}{\sigma}$

$\dfrac{\sigma}{\varepsilon\omega}$ is called loss tangent

$\tau \ll \dfrac{1}{\omega} \quad \left(\dfrac{\sigma}{\varepsilon\omega} \gg 1\right)$ Good conductor

$\tau \gg \dfrac{1}{\omega} \quad \left(\dfrac{\sigma}{\varepsilon\omega} \ll 1\right)$ Poor conductor

For metals
$\rho \sim 10^{-7}\ \Omega\,\mathrm{m}$

$\tau \sim 10^{-19}$ s

The accumulated free charges eventually dissipate after *t*. In this case, one has

$$\rho_f = 0$$

Wave equations for **E** and **B**

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \left(\nabla \times \mathbf{E}\right) = \nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2\mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{B}\right) = -\mu\sigma\frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\boxed{\begin{aligned} \nabla^2\mathbf{E} &= \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{E}}{\partial t} \\ \nabla^2\mathbf{B} &= \mu\varepsilon\frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{B}}{\partial t} \end{aligned}}$$

$$\nabla \times \left(\nabla \times \mathbf{B}\right) = \nabla\left(\nabla \cdot \mathbf{B}\right) - \nabla^2\mathbf{B} = \nabla \times \left(\mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= \mu\sigma\left(\nabla \times \mathbf{E}\right) + \mu\varepsilon\frac{\partial}{\partial t}\left(\nabla \times \mathbf{E}\right) = -\mu\sigma\frac{\partial \mathbf{B}}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

The attenuated plane waves

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z - \omega t)}$$

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$\tilde{k} = k + i\kappa$$

$$k = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$\kappa = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$$

The attenuated plane waves

Low-loss dielectric (poor conductors)

For small loss tangent $\frac{\sigma}{\varepsilon\omega} \ll 1$

$$\kappa = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2} \simeq \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$k = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2} \simeq \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{2}$$

$$= \omega\sqrt{\mu\varepsilon}$$

Skin depth

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad \Longrightarrow \quad d = \frac{1}{\kappa}$$

A characteristic attenuation length: skin depth

The real part , *k* represents a propagating EM wave with

$$\lambda = \frac{2\pi}{k} \qquad v = \frac{\omega}{k}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \quad \Longrightarrow$$

The attenuated plane waves are also transverse polarized

Phase lag of the **B** field

Re-orient the axes so that **E** is polarized along *x* direction

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{\mathbf{x}} \qquad \tilde{\mathbf{B}}(z,t) = \frac{1}{\omega} \tilde{\mathbf{k}} \times \tilde{\mathbf{E}}(z,t)$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{\mathbf{y}} = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{\mathbf{y}}$$

In principle, the wave vector can be expressed by $\tilde{k} = k + i\kappa = Ke^{i\phi}$

$$K = \sqrt{k^2 + \kappa^2} \qquad \tan\phi = \frac{\kappa}{k}$$

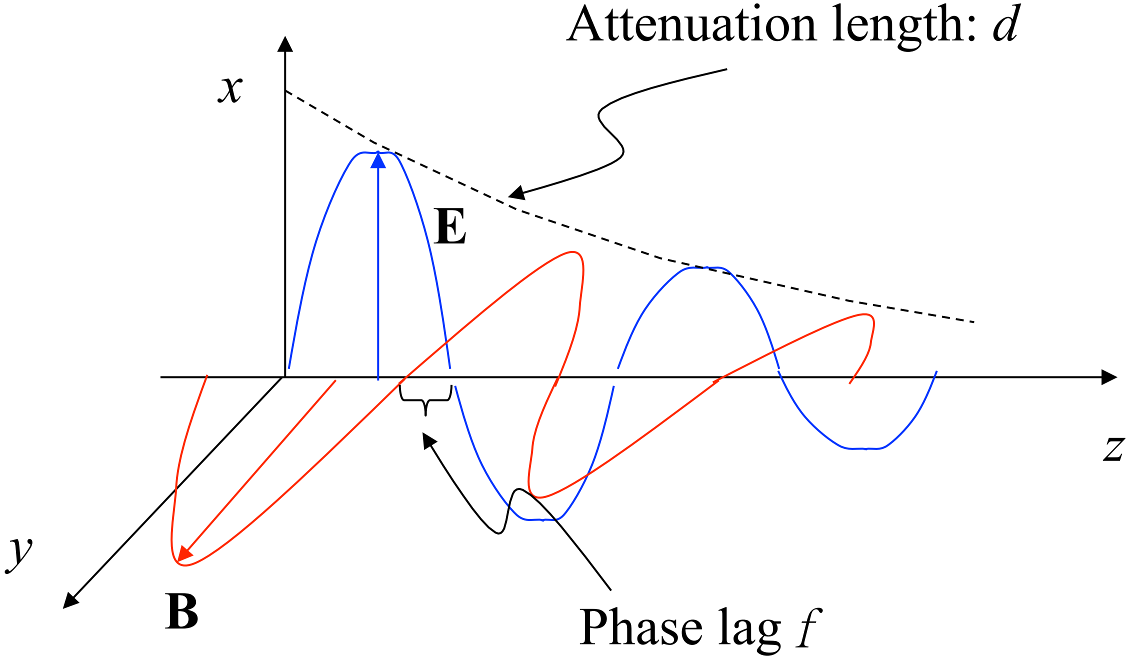
Suppose the complex numbers $\tilde{E}_0 = E_0 e^{i\delta_E}$ $\tilde{B}_0 = B_0 e^{i\delta_B}$

The two phase constants obey $\delta_E - \delta_B = \phi$

The attenuated EM waves in conductor

The field amplitudes obeys $\frac{B_0}{E_0} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \right]^{1/2}$

A phase lag (in space) for B field $\phi = \tan^{-1}\frac{\kappa}{k}$



Good conductors

$$\frac{\sigma}{\varepsilon\omega} \gg 1 \qquad \kappa = k \simeq \sqrt{\frac{\mu\omega\sigma}{2}} \qquad \phi = \frac{\pi}{4}$$

The wave number becomes much larger

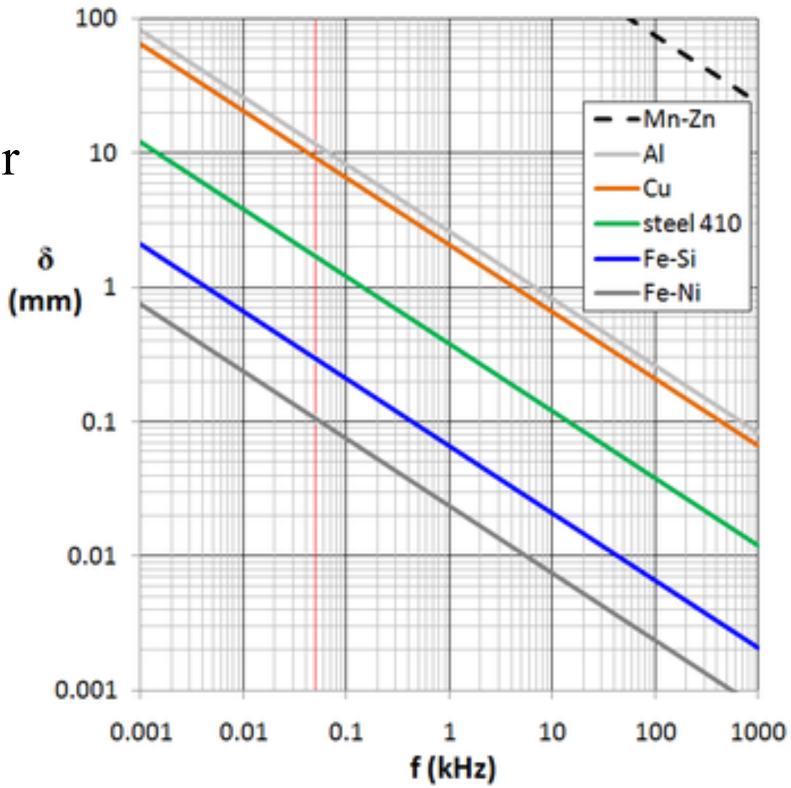
A smaller wavelength in a good conductor

$\mu_0 \sim 10^{-6}$ H/m

$\sigma \sim 10^{7}$ S m^{-1}

Optical $\omega \sim 10^{15}$ Hz

Skin depth d $\sim 10^{-8}$ m

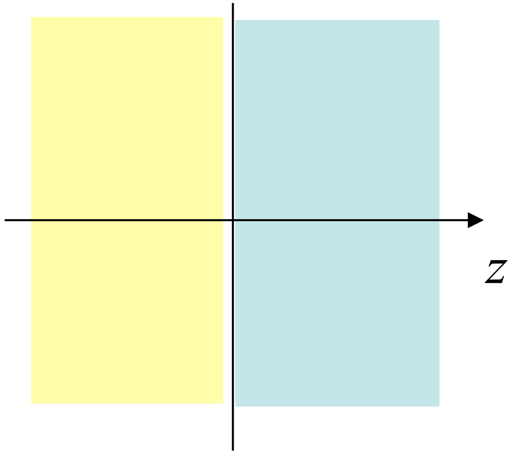


Reflection at a conducting surface

Incident waves

$$\tilde{\mathbf{E}}_I\left(z,t\right)=\tilde{E}_{0,I}e^{i\left(k_1 z-\omega t\right)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_I\left(z,t\right)=\frac{1}{v_1}\tilde{E}_{0,I}e^{i\left(k_1 z-\omega t\right)}\hat{\mathbf{y}}$$



Non-conducting
linear media

conductor

Reflected waves

$$\tilde{\mathbf{E}}_R\left(z,t\right)=\tilde{E}_{0,R}e^{i\left(-k_1 z-\omega t\right)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_R\left(z,t\right)=-\frac{1}{v_1}\tilde{E}_{0,R}e^{i\left(-k_1 z-\omega t\right)}\hat{\mathbf{y}}$$

Transmitted waves

$$\tilde{\mathbf{E}}_T\left(z,t\right)=\tilde{E}_{0,T}e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_T\left(z,t\right)=\frac{\tilde{k}_2}{\omega}\tilde{E}_{0,T}e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{\mathbf{y}}$$

boundary conditions

Recall the general boundary conditions

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \quad \Longrightarrow \quad \sigma_f = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \qquad \tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}$$

$$\frac{\mathbf{B}_1^{\parallel}}{\mu_1} - \frac{\mathbf{B}_2^{\parallel}}{\mu_2} = \mathbf{K}_f \times \hat{\mathbf{n}} \qquad \frac{1}{\mu_1 v_1}\left(\tilde{E}_{0,I} - \tilde{E}_{0,R}\right) = \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0,T} \qquad \mathbf{K}_f = 0$$

$$\tilde{E}_{0,R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0,I} \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

$$\tilde{E}_{0,T} = \frac{2}{1+\tilde{\beta}}\tilde{E}_{0,I}$$

Reflection on a perfect conductor

For a perfect conductor, $\sigma \to \infty$

$$k_2 \to \infty$$

$$\tilde{\beta} \to \infty$$

⇨

$$\tilde{E}_{0,R} \simeq -\tilde{E}_{0,I}$$

$$\tilde{E}_{0,T} \simeq 0$$

Total reflection with a 180^{o} phase shift

For silver, the skin depth is on the order of 10nm at optical frequencies

The frequency dependence of permittivity

Dispersion: the refraction index n depends on wavelength

If the speed of wave depends on its frequency, the medium is called **dispersive**

Wave (phase) velocity: the speed at which a sinusoidal component travels

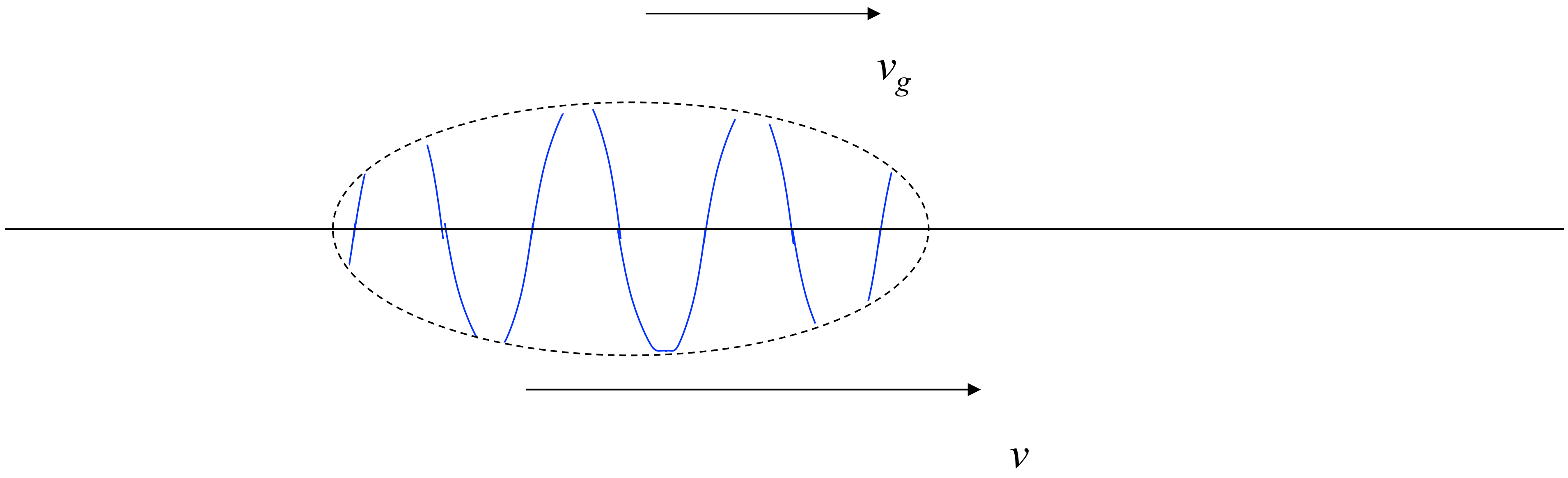
$$v = \frac{\omega}{k}$$

group velocity: the speed at which a wave packet travels

$$v_g = \frac{d\omega}{dk}$$

The surface waves on water have a v two times of v_g

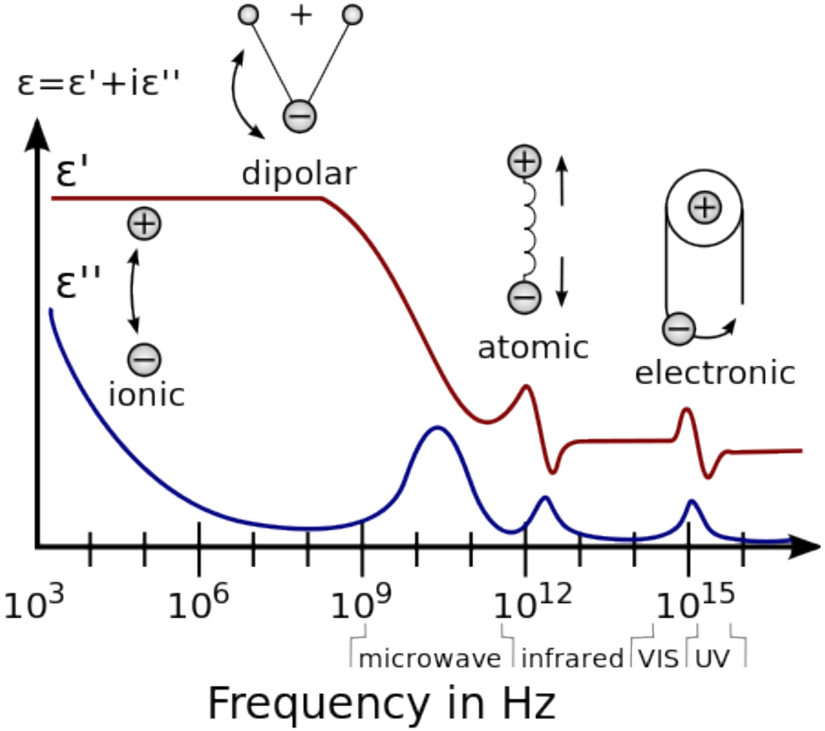
Phase velocity vs group velocity

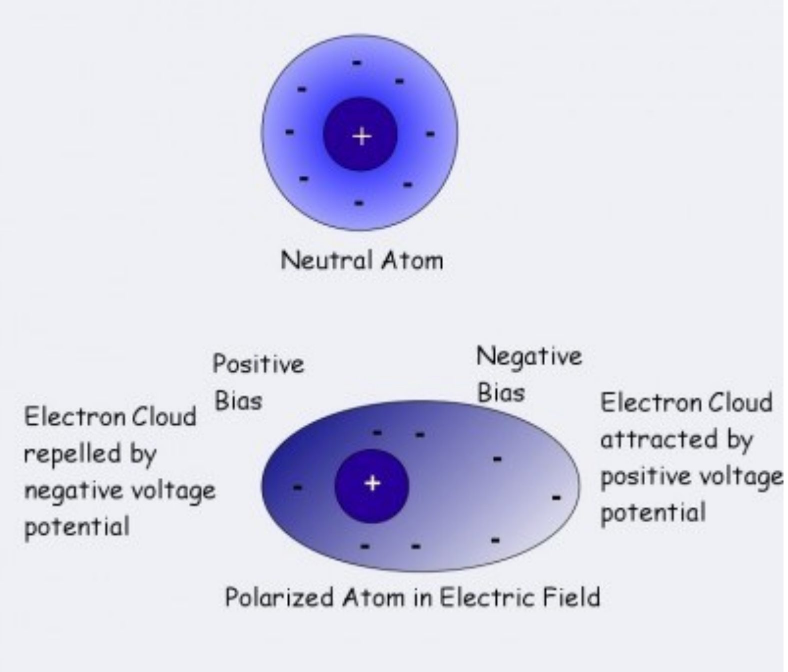


The energy carried by a wave packet ordinarily travels at the group velocity

In some circumstances, phase velocity is larger than *c*

Atomic polarization



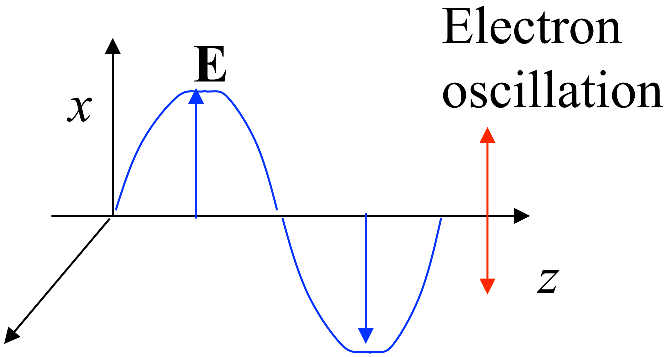


AC polarization in dielectrics

The electrons are bounded to specific molecules. The binding force can be modeled obeying Hooke's law:

$$F_{binding} = -k_s x = -m\omega_0^2 x$$

ω_0 Natural frequency



There is some damping on the electron oscillator:

$$F_{damping} = -m\gamma \frac{dx}{dt}$$

The presence of an EM wave yields the electron a driving force

$$F_{driving} = qE = qE_0 \cos \omega t$$

$q = -e$ for electrons

Damped oscillator

The Newton' s 2nd law asserts that

$$m\frac{d^2x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$$

Namely,

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE\cos\omega t$$

The equation can be re-casted in a complex form:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}Ee^{-i\omega t}$$

The steady state solution is simply $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$

Damped oscillator

The amplitude follows that

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0$$

The induced dipole moment is the real part of

$$\tilde{p} = q\tilde{x}_0 = \boxed{\frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}} E_0 e^{-i\omega t}$$

The argument in this complex results in a phase lag of p to E

$$\text{phase} = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = \begin{cases} 0 & \omega \ll \omega_0 \\ \pi & \omega \gg \omega_0 \end{cases}$$

Complex susceptibility

In general, differently situated electrons in a molecule experience different natural frequencies and damping coefficients

If assuming f_j electrons with frequency w_j and damping g_j in each molecule, one has the polarization given by

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}}$$

or expressed by a complex susceptibility: $\tilde{\mathbf{P}} = \varepsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}}$

The susceptibility yields a complex dielectric constant:

$$\tilde{\varepsilon}_r = 1 + \frac{Nq^2}{m\varepsilon_0}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

EM waves in a dispersive medium

The complex electric field obeys the wave equation that

$$\nabla^2 \tilde{\mathbf{E}} = \mu_0 \tilde{\varepsilon} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}$$

Again, the plane wave solutions are

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}$$

With a complex wave number $\tilde{k} = \sqrt{\tilde{\varepsilon}\mu_0}\,\omega$

expressing the wave number as $\tilde{k} = k + i\kappa$

k describes the attenuation of EM waves in the medium. The intensity loss per unit travel distance is the **absorption coefficient** $\alpha = 2\kappa$

k describes the propagation of the EM wave, and relates to the index of refraction as $n = ck/\omega$

Anomalous dispersion

$$\tilde{\varepsilon}_r = 1 + \frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

For gases, the second term is small, so the square root of ε_r can be approximated by

$$\sqrt{\tilde{\varepsilon}_r} \simeq 1 + \frac{1}{2}\frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

Therefore the complex wave number becomes

$$\tilde{k} = \frac{\omega}{c}\left[1 + \frac{1}{2}\frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

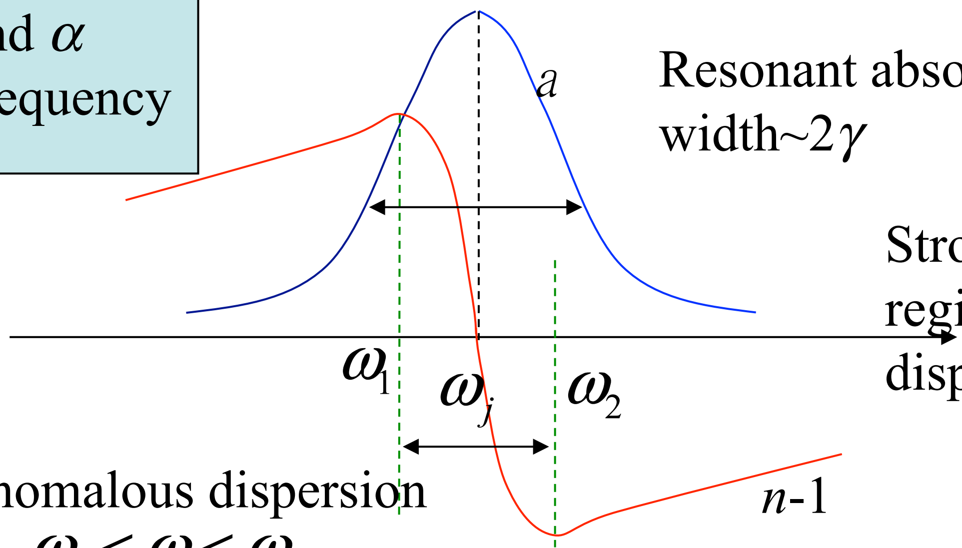
Anomalous dispersion

In turn, one has

$$n = 1 + \mathrm{Re}\frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = 1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}$$

$$\alpha = 2\kappa = \mathrm{Im}\frac{Nq^2\omega}{m\varepsilon_0 c}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = \frac{Nq^2\omega^2}{m\varepsilon_0 c}\sum_j \frac{f_j\gamma_j^2}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}$$

The plot of *n*-1 and α near a resonant frequency



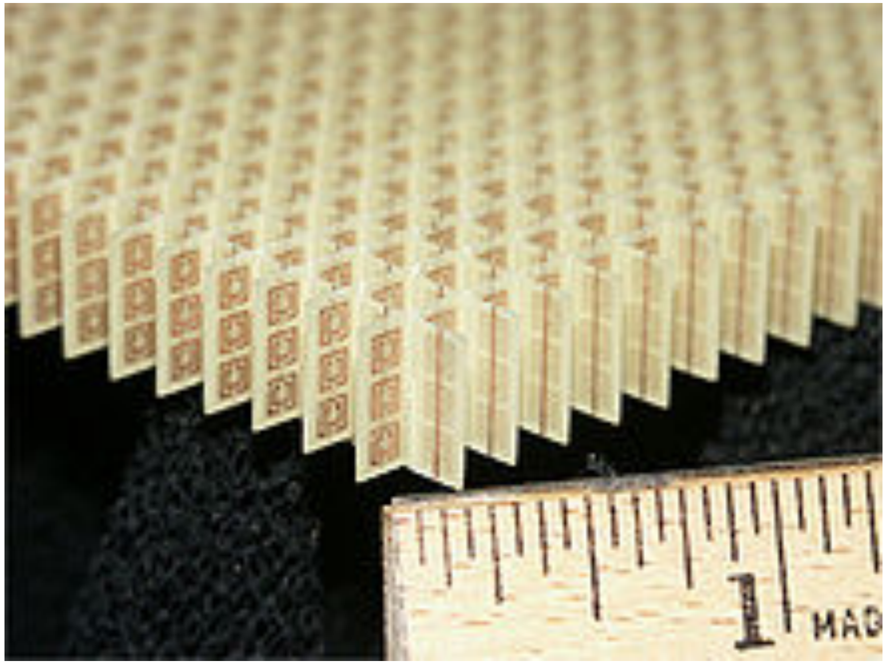
Resonant absorption width~2γ

Strong absorption in the region of anomalous dispersion

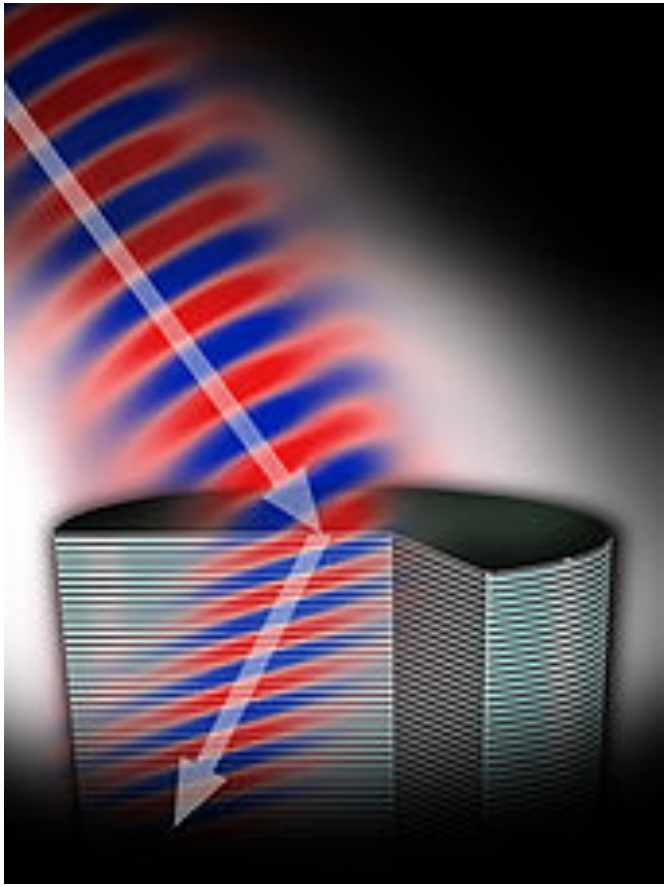
Anomalous dispersion

$$\omega_1 < \omega < \omega_2$$

Negative index meta materials



Electric dipole resonance
Magnetic dipole resonance



The low frequency result

When the EM wave frequency is much smaller than the resonant frequencies, one has.

$$n = 1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j\left(\omega_j^2-\omega^2\right)}{\left(\omega_j^2-\omega^2\right)^2+\gamma_j^2\omega^2}$$

$$\simeq 1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2} \simeq 1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right)$$

$$= 1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2} + \frac{Nq^2\omega^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^4}$$

(For instance the transparent materials have the lowest significant resonances at UV frequencies.)

Cauchy' s formula $n = 1 + A\left(1+\frac{B}{\lambda^2}\right)$