Spin

2018/4/30

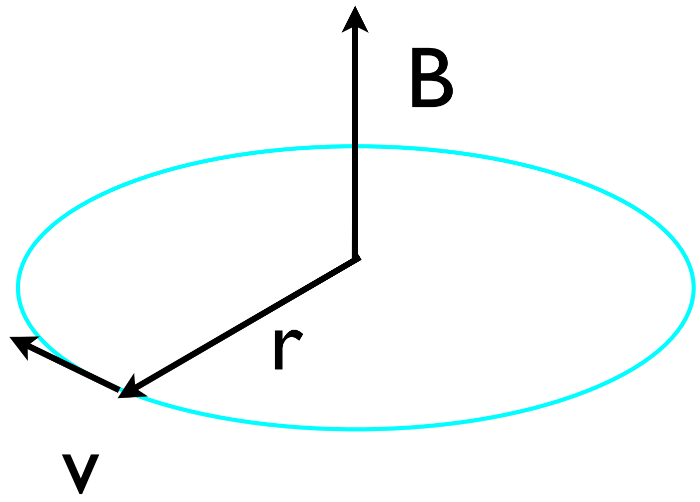
Orbital magnetic dipole moment

- for an electron moving in a circular orbit

$$i = \frac{e}{T} = \frac{ev}{2\pi r}$$

- in classical electrodynamics, it produces a magnetic dipole moment

$$\mu_l = iA = \frac{ev}{2\pi r}\pi r^2 = \frac{evr}{2}$$



Bohr magneton

- The electron also has an angular momentum

$$L = mvr$$

- The dipole moment and L are related to each other

$$\frac{\mu}{L} = \frac{evr/2}{mvr} = \frac{e}{2m} = \frac{g_l \mu_b}{\hbar}$$

- A constant Bohr magneton is defined

$$\mu_b = \frac{e\hbar}{2m} = 0.927 \times 10^{-23} \text{ A m}^2$$

Gyromagnetic ratio

- The constant g_l is called gyromagnetc ratio
- For orbital motion $g_l = 1$
- The magnetic dipole moment can be written as

$$\mu = -\frac{g_l \mu_b}{\hbar} L$$

- The dipole moment and L are in anti-parallel because of negative charge

Quantum results

- for angular momentum eigenstates

$$L = l(l+1)\hbar \qquad\qquad L_z = m\hbar$$

- The dipole moment has

$$\mu_l = \sqrt{l(l+1)}g_l\mu_b$$

$$\mu_{l,z} = -mg_l\mu_b$$

Energy in a magnetic field

- A magnetic dipole moment experiences a torque in a magnetic field

$$\vec{\tau} = \vec{\mu}_l \times \vec{B}$$

- The force is conservative, and gives a potential energy

$$\Delta E = -\vec{\mu}_l \cdot \vec{B}$$

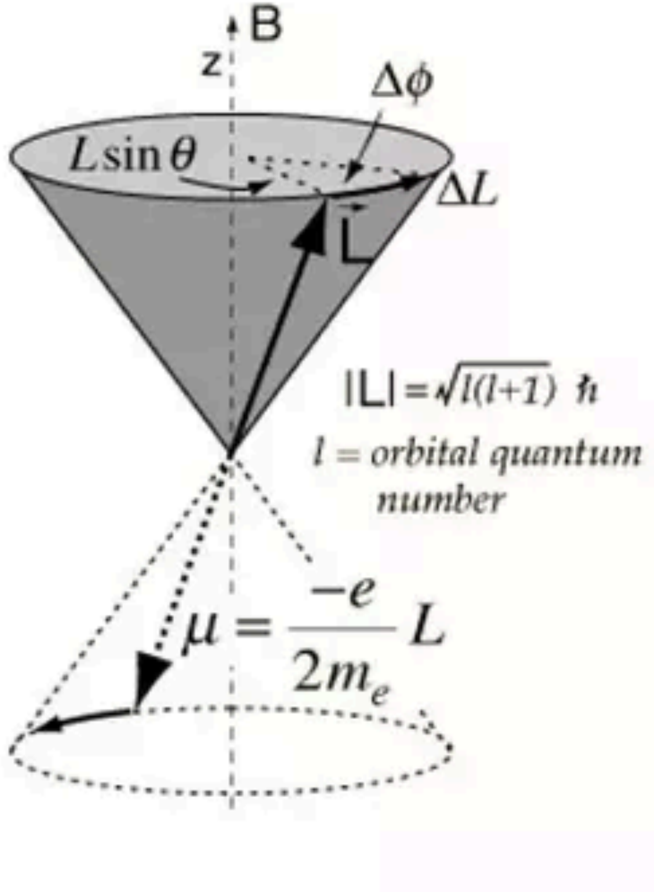
precession

- The torque produces a change in angular momentum

$$\tau = \mu_l B \sin\theta$$

$$\frac{\Delta L}{\Delta t} = \tau = \mu_l B \sin\theta = \frac{g_l \mu_b}{\hbar} L B \sin\theta$$

$$\frac{\Delta L}{\Delta t} = \frac{g_l \mu_b}{\hbar} B L_\perp$$



- Precession angle is

$$\Delta\phi = \frac{\Delta L}{L_\perp} = \frac{g_l \mu_b}{\hbar} B \Delta t \qquad \Omega = \frac{\Delta\phi}{\Delta t} = \frac{g_l \mu_b}{\hbar} B$$

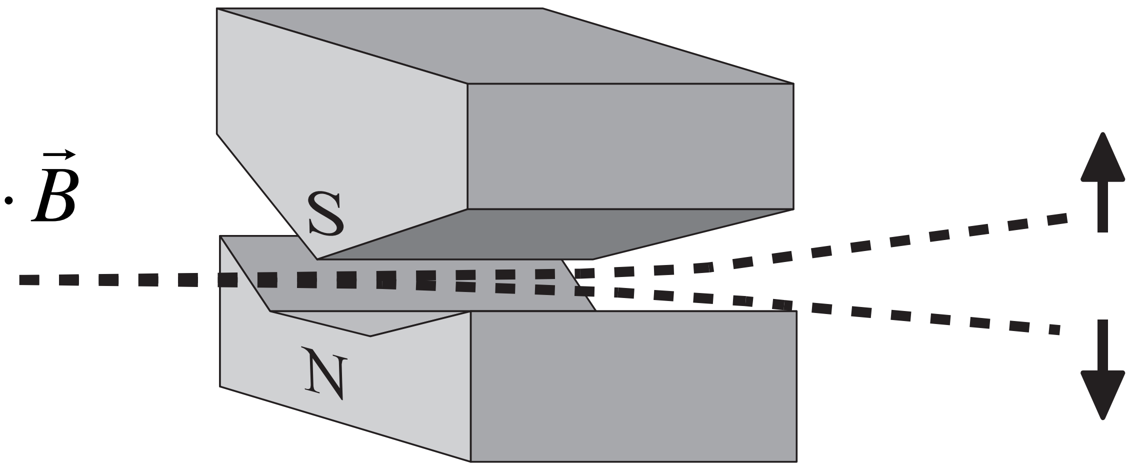
Stern-Gerlach experiment

- A stream of atoms moving from the right passes between the asymmetric poles of a magnet. Particles with different values of μ_z are deflected in different directions. The final position of the atom determines its μ_z

γ is gyromagnetic ratio

$$\vec{\mu} \propto -\vec{S}$$

$$E = -\vec{\mu} \cdot \vec{B}$$



spin 1/2 system

- A particle may have an intrinsic angular momentum called spin
- Electrons, protons, and neutrons are all examples of spin-1/2 particles
- If one measure the z-component S_z(or S_x, S_y) of the spin angular momentum for one of these particles, he gets

$$S_z = \pm\frac{\hbar}{2}$$

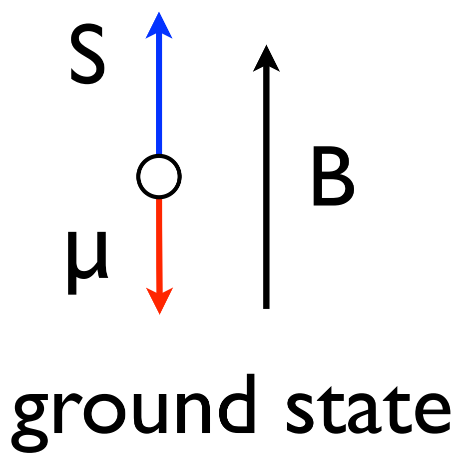
intrinsic magnetic moment

- electron has an intrinsic magnetic dipole moment by virtue of its spin

$$\mu = -\frac{g_s \mu_b}{\hbar}\mathbf{S}$$

- gyromagnetic ratio, g_s=2
- Hamiltonian

$$H = -\mu \cdot \mathbf{B} = -\frac{g_s \mu_b}{\hbar}\mathbf{S}\cdot\mathbf{B}$$



Spin 1/2 system

- for angular momentum S=1/2, there are 2 eigenstates

$$s = \frac{1}{2} \qquad m_s = \pm\frac{1}{2}$$

- we can write the states

$$S^2\chi_\pm = s(s+1)\hbar^2$$

$$S_z\chi_\pm = \pm\frac{1}{2}\hbar$$

commutation relations

- mutual commutation relations for L

$$\left[L_x, L_y\right] = \left[yp_z - zp_y, zp_x - xp_z\right] = \left[yp_z, zp_x\right] + \left[zp_y, xp_z\right]$$

$$= -i\hbar y p_x + i\hbar p_y x = i\hbar L_z$$

$$\left[L_y, L_z\right] = i\hbar L_x$$

$$\left[L_z, L_x\right] = i\hbar L_y$$

- mutual commutation relations for S

$$\left[S_x, S_y\right] = i\hbar S_z$$

$$\left[S_y, S_z\right] = i\hbar S_x$$

$$\left[S_z, S_x\right] = i\hbar S_y$$

raising and lowering operators

- to show the structure, we define $S_{\pm} = S_x \pm iS_y$

$$\left[S^2, S_{\pm}\right] = \left[S^2, S_x\right] \pm i\left[S^2, S_y\right] = 0$$

$$\left[S_z, S_{\pm}\right] = \left[S_z, S_x\right] \pm i\left[S_z, S_y\right]$$

$$= i\hbar S_y \pm \hbar S_x = \pm\hbar S_{\pm}$$

- total angular momentum does not change

$$S^2\left(S_+\chi_{\pm}\right) = \left(S^2 S_+\right)\chi_{\pm}$$

$$= \left(S_+ S^2\right)\chi_{\pm} + \left[S^2, S_+\right]\chi_{\pm}$$

$$= s\left(s+1\right)\hbar^2\left(S_+\chi_{\pm}\right)$$

meaning of S+

- z-component

$$S_z\left(S_+\chi_-\right)=\left(S_zS_+\right)\chi_-$$
$$=\left(S_+S_z\right)\chi_-+\left[S_z,S_+\right]\chi_-$$
$$=-\frac{1}{2}\hbar\left(S_+\chi_-\right)+\hbar\left(S_+\chi_-\right)=\frac{1}{2}\hbar\left(S_+\chi_-\right)$$

$$S_+\chi_-=C_+\chi_+$$

$$S_z\left(S_+\chi_+\right)=\left(S_zS_+\right)\chi_+$$
$$=\left(S_+S_z\right)\chi_++\left[S_z,S_+\right]\chi_+$$
$$=\frac{1}{2}\hbar\left(S_+\chi_+\right)+\hbar\left(S_+\chi_+\right)=\frac{3}{2}\hbar\left(S_+\chi_+\right)$$

$$S_+\chi_+=0$$

- because $\left\langle S_z^2\right\rangle$ must be smaller than $\left\langle S^2\right\rangle$

meaning of S-

- z-component

$$S_z(S_-\chi_-) = (S_z S_-)\chi_-$$
$$= (S_- S_z)\chi_- + [S_z, S_-]\chi_-$$
$$= -\frac{1}{2}\hbar(S_-\chi_-) - \hbar(S_-\chi_-) = -\frac{3}{2}\hbar(S_-\chi_-)$$

$$S_-\chi_- = 0$$

$$S_z(S_-\chi_+) = (S_z S_-)\chi_+$$
$$= (S_- S_z)\chi_+ + [S_z, S_-]\chi_+$$
$$= \frac{1}{2}\hbar(S_-\chi_+) - \hbar(S_-\chi_+) = -\frac{1}{2}\hbar(S_-\chi_+)$$

$$S_-\chi_+ = C_-\chi_-$$

eigenstate: spinor

- spinor state(we cannot find spatial functions for them)

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- any spinor state(normalized)

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\chi_+ + \beta\chi_-$$

$$1 = \langle \chi | \chi \rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2$$

↑

Dirac notation

Operators

- Because of the properties, we can write the operators

$$S^2 = \begin{pmatrix} \frac{3}{4}\hbar & 0 \\ 0 & \frac{3}{4}\hbar \end{pmatrix} = \frac{3}{4}\hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_z = \begin{pmatrix} \frac{1}{2}\hbar & 0 \\ 0 & -\frac{1}{2}\hbar \end{pmatrix} = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_+ = C_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = C_- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Sx and Sy

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}\begin{pmatrix} 0 & C_+ \\ C_- & 0 \end{pmatrix} \qquad S_x = \frac{1}{2i}(S_+ - S_-) = \frac{1}{2i}\begin{pmatrix} 0 & C_+ \\ -C_- & 0 \end{pmatrix}$$

- The hermicitivity of Sx and Sy gives

$$C_+ = C_-^*$$

- We choose C's are real. Notice that the eigenvalues of Sx and Sy are $\pm\frac{1}{2}\hbar$

$$C_+ = C_- = \hbar \qquad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrices

- Hermitian operators in 2 level systems $\mathbf{S} = \frac{1}{2}\hbar\sigma$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Commutation relations

$$\left[\sigma_x, \sigma_y\right] = 2i\sigma_z \qquad \left[S_x, S_y\right] = i\hbar S_z$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

- They are anti-commute

$$\left\{\sigma_a, \sigma_b\right\} = 2\delta_{ab}$$

eigenstates of S_x

- To find the eigenstates for $$S_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The eigenequation $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}$$

- The eigenevalue $\lambda^2 - 1 = 0$ $\lambda = \pm 1$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Bloch sphere

eigenstates of Sz

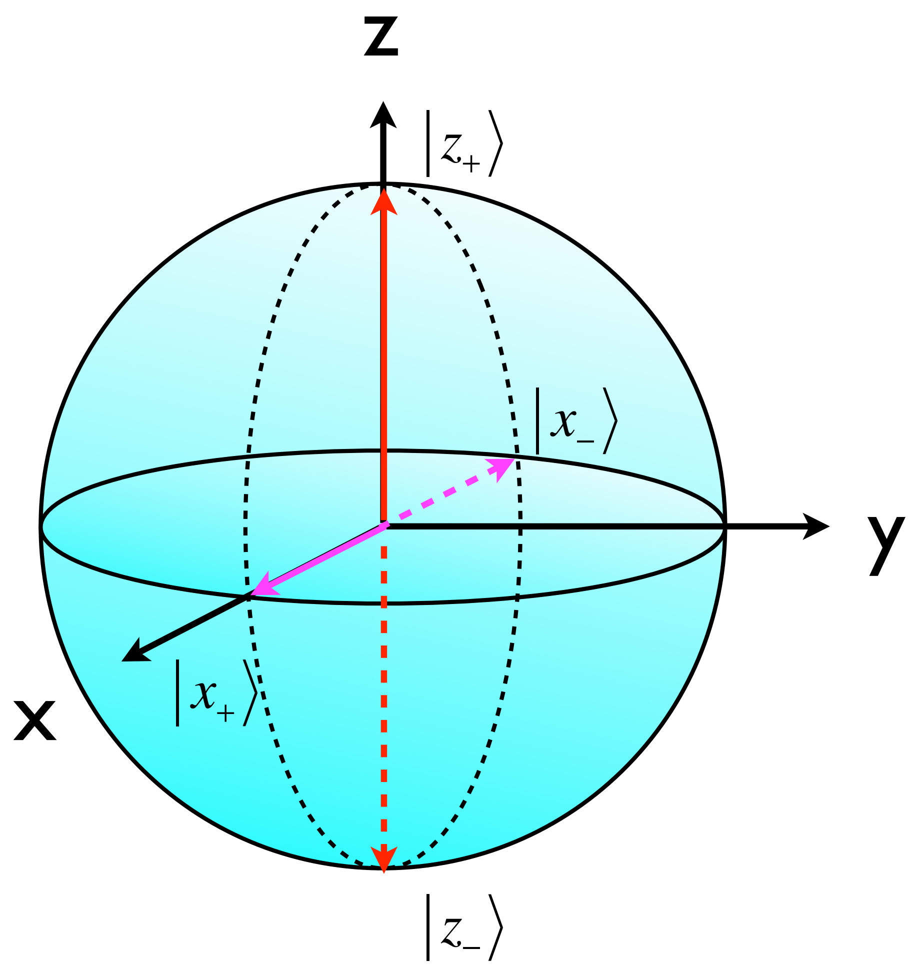
$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

eigenstates of Sx

$$|x_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{1}{\sqrt{2}}|z_-\rangle$$

$$|x_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{1}{\sqrt{2}}|z_-\rangle$$

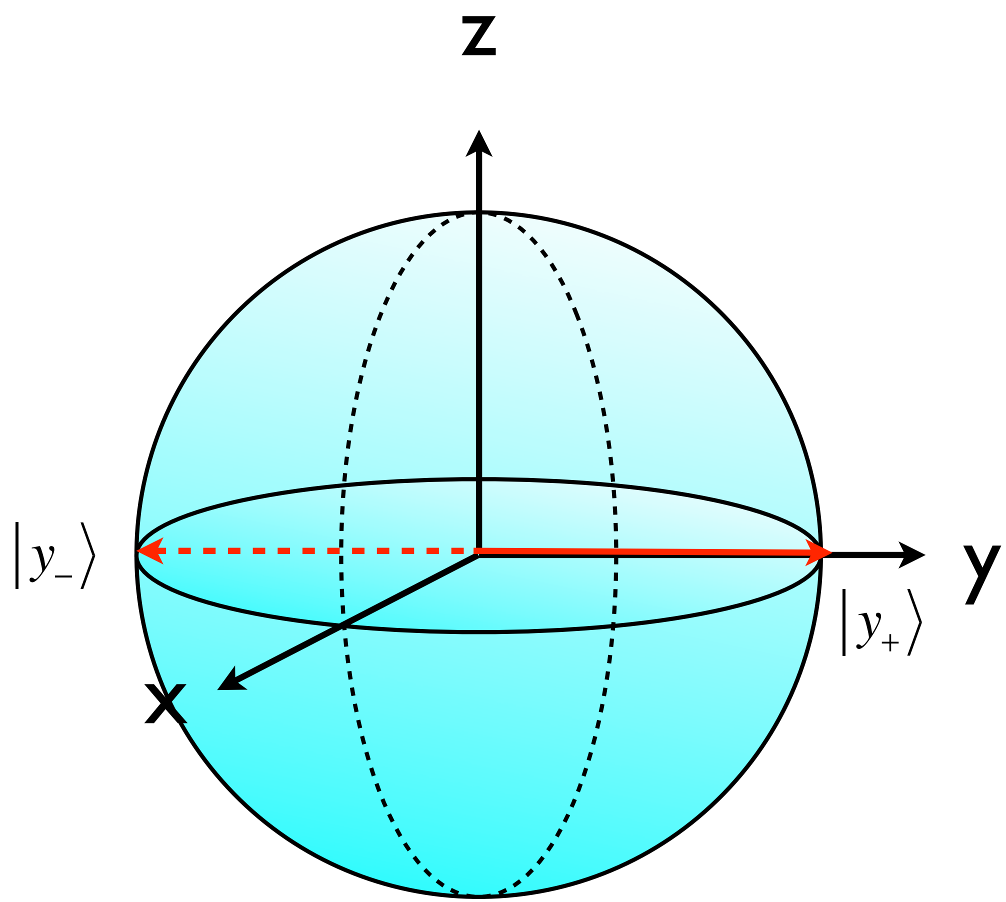


Bloch sphere

eigenstates of Sy

$$|y_+\rangle = \frac{1}{\sqrt{2}}|z_+\rangle + \frac{i}{\sqrt{2}}|z_-\rangle$$

$$|y_-\rangle = \frac{1}{\sqrt{2}}|z_+\rangle - \frac{i}{\sqrt{2}}|z_-\rangle$$



some eigenstates

- To find the eigenstates for

$$S_\theta = S_z \cos\theta + S_x \sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

- The eigenequation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

- The eigenevalue $\lambda^2 - 1 = 0$ $\lambda = \pm 1$

- for $\lambda = 1$ $\cos\theta u + \sin\theta v = u$

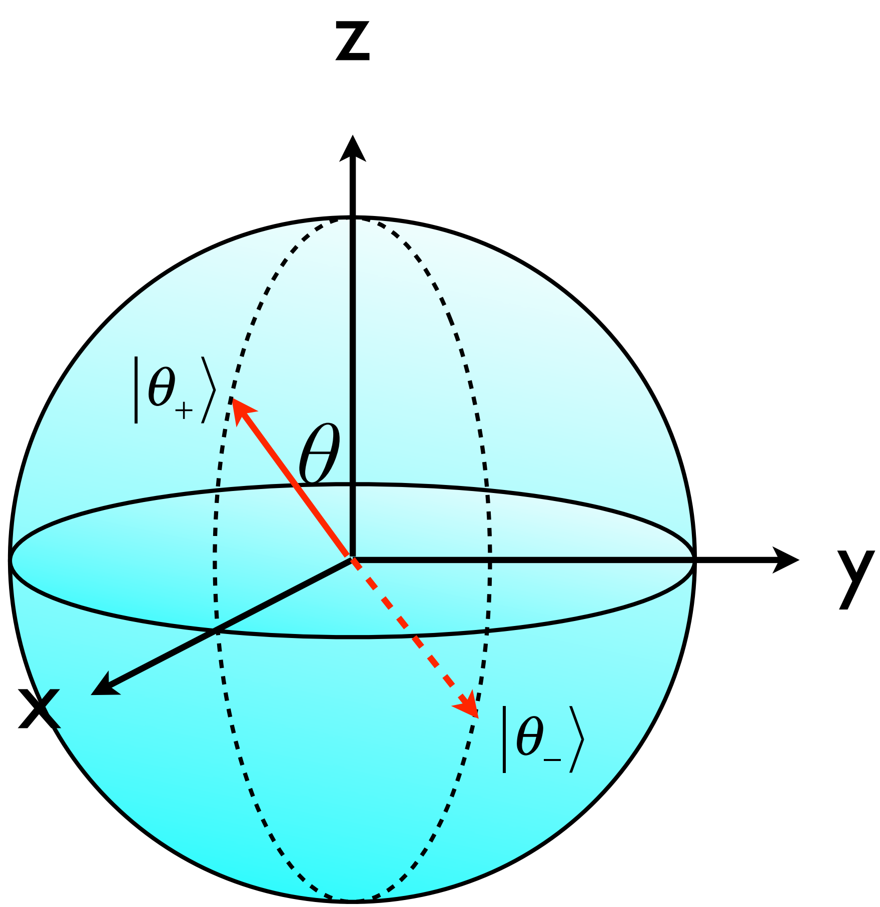
$$|\theta_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}|z_+\rangle + \sin\frac{\theta}{2}|z_-\rangle \quad |\theta_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} = \sin\frac{\theta}{2}|z_+\rangle - \cos\frac{\theta}{2}|z_-\rangle$$

rotation in θ

- Suppose we choose a direction in the xz-plane that is inclined at an angle θ from the z-axis. Then the amplitude vectors

$$|\theta_+\rangle = \cos\frac{\theta}{2}|z_+\rangle + \sin\frac{\theta}{2}|z_-\rangle$$

$$|\theta_-\rangle = \sin\frac{\theta}{2}|z_+\rangle - \cos\frac{\theta}{2}|z_-\rangle$$



more eigenstates

- To find the eigenstates for

$$S_\phi = S_x \cos\phi + S_y \sin\phi = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

- The eigenequation

$$\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

- The eigenevalue $\lambda^2 - 1 = 0$ $\lambda = \pm 1$

- for $\lambda = 1$ $u = e^{-i\phi} v$ $\lambda = -1$ $u = -e^{-i\phi} v$

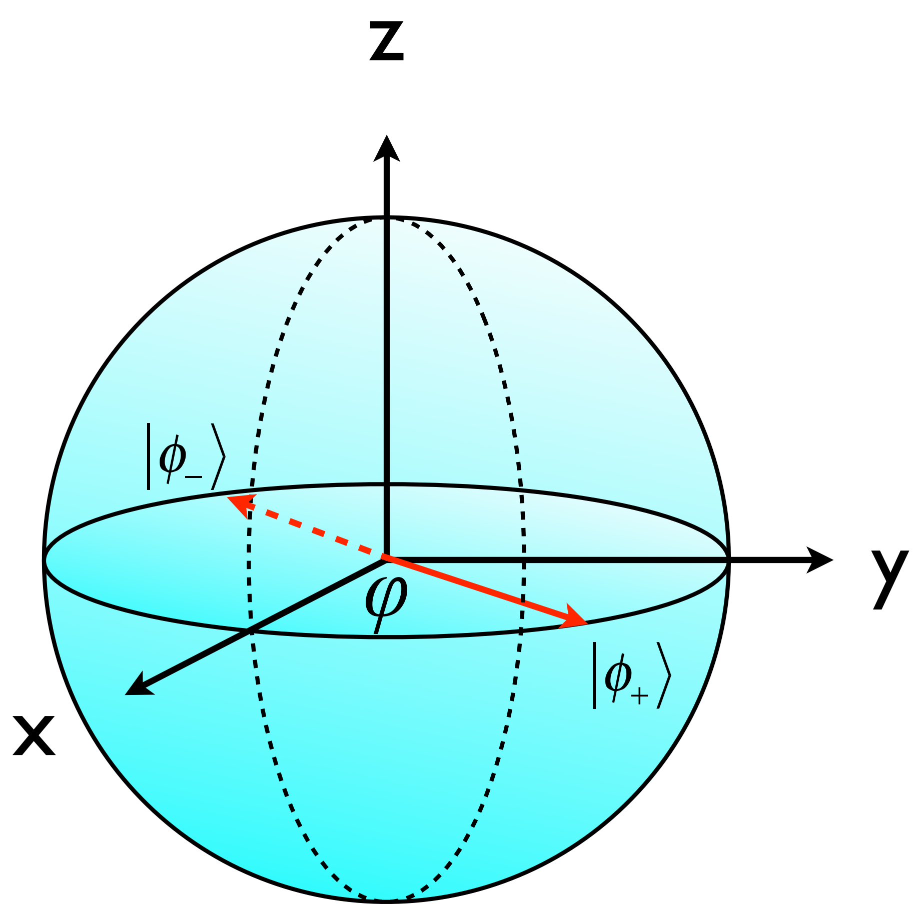
$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \qquad |\phi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{-i\phi} \end{pmatrix}$$

rotation in φ

eigenstates of S_ϕ

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$$

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix}$$

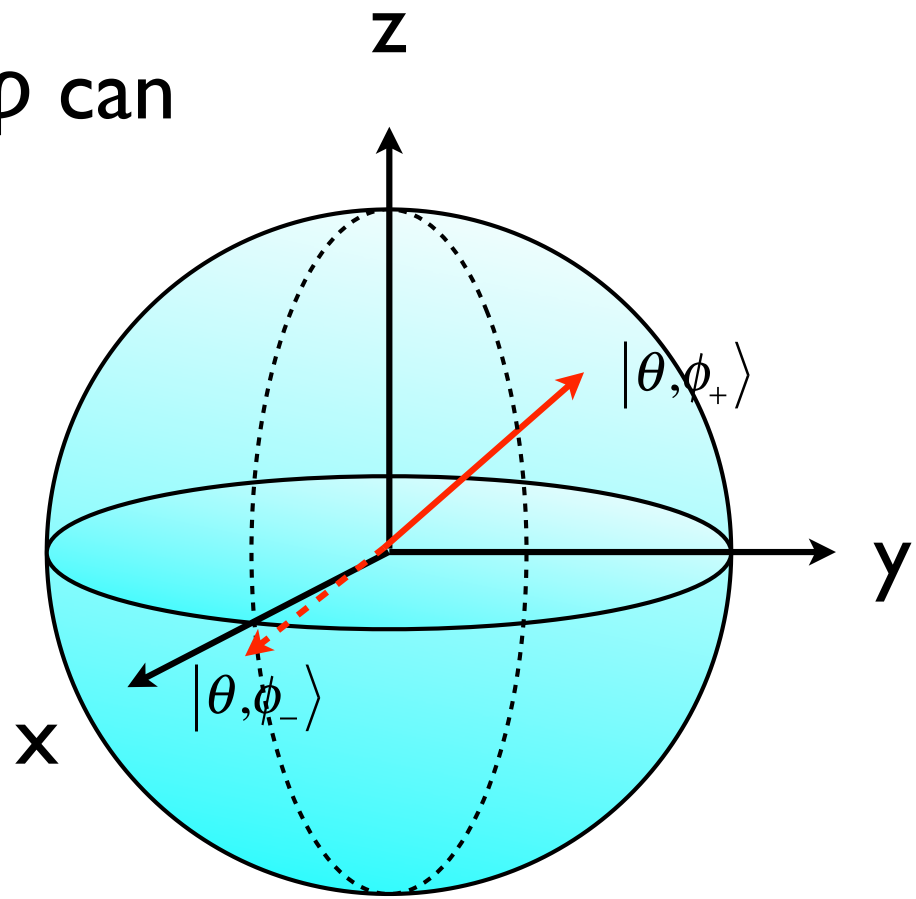


General case

- Any rotation in θ and φ can be shown that

$$|\theta,\phi_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$|\theta,\phi_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{-i\phi} \end{pmatrix}$$



are eigenstates of

$$S_{\theta,\phi} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_x \cos\theta = \mathbf{n}_{\theta,\phi} \cdot \mathbf{S}$$

expectation values

$$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = \frac{1}{2}\hbar \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{1}{2}\hbar \left(\alpha^* \beta + \beta^* \alpha \right)$$

$$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = \frac{1}{2}\hbar \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= -\frac{i}{2}\hbar \left(\alpha^* \beta - \beta^* \alpha \right)$$

$$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = \frac{1}{2}\hbar \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{1}{2}\hbar \left(|\alpha|^2 - |\beta|^2 \right)$$

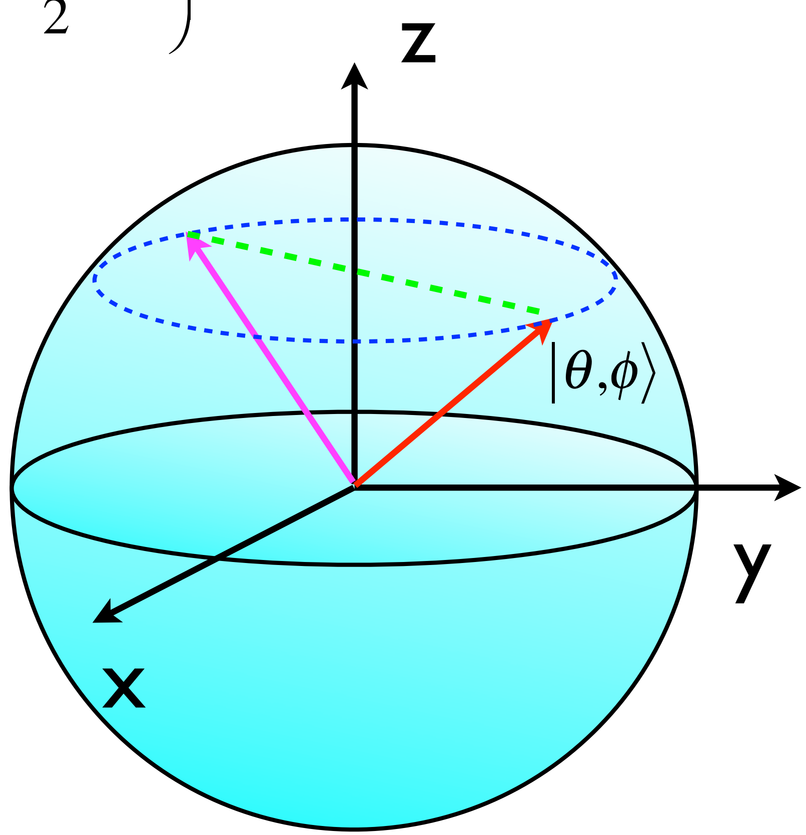
rotation about z

$$\sigma_z|\theta,\phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$\theta \to \theta$$
$$\phi \to \pi + \phi$$

can be viewed as the rotation about z of π

also called Pauli-Z gate



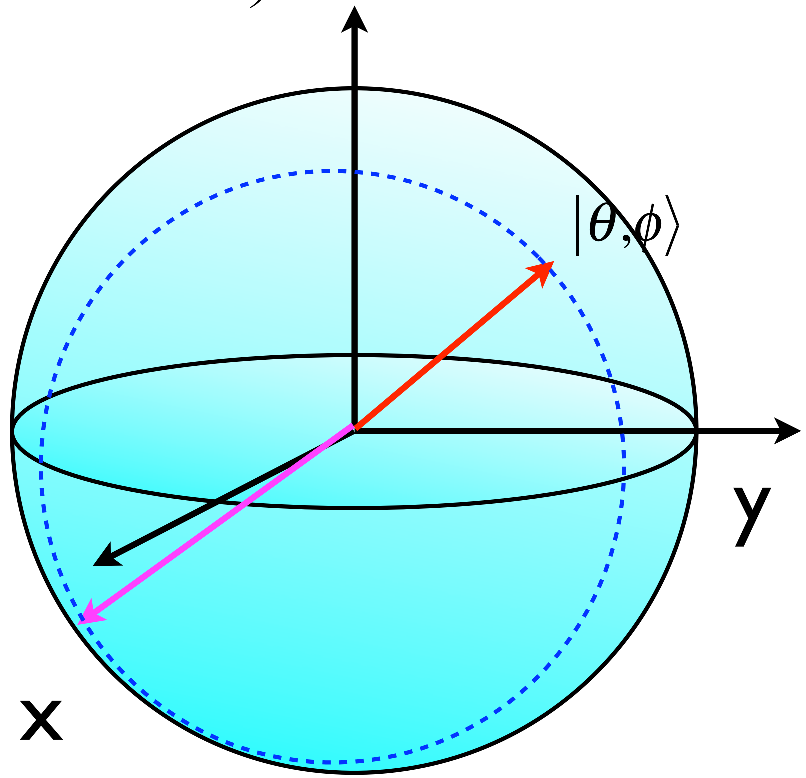
rotation about x

$$\sigma_x|\theta,\phi_+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} = e^{i\phi} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} e^{-i\phi} \end{pmatrix}$$

$$\theta \to \pi - \theta$$
$$\phi \to -\phi$$

rotation about x of π

also called Pauli-X gate
or NOT gate



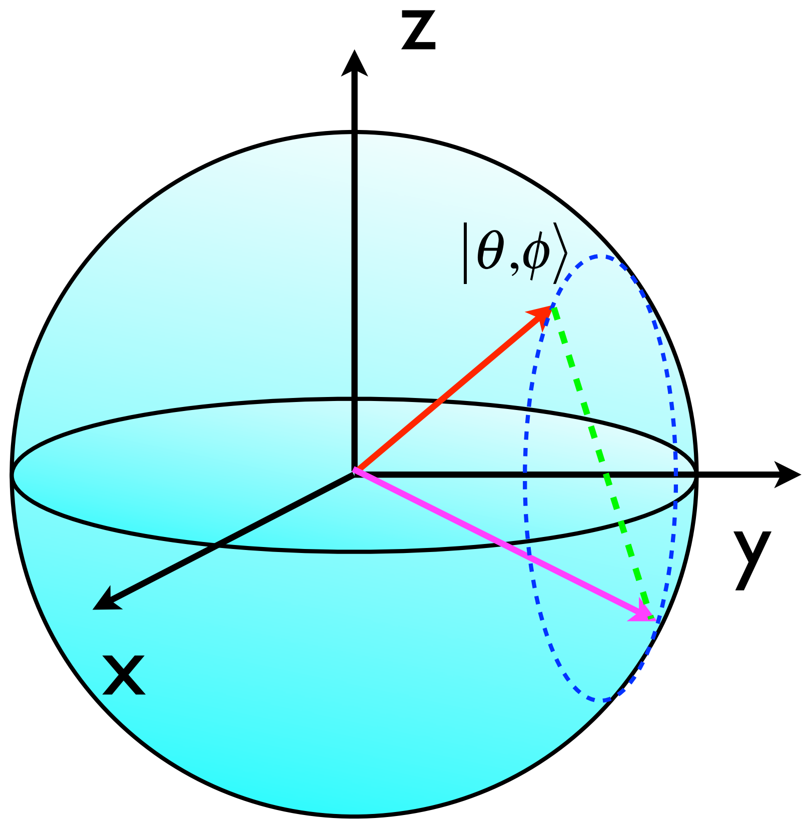
rotate about y

$$\sigma_y|\theta,\phi_+\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} = \begin{pmatrix} -i\sin\frac{\theta}{2} e^{i\phi} \\ i\cos\frac{\theta}{2} \end{pmatrix} = -ie^{i\phi} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{-i\phi} \end{pmatrix}$$

$$\theta \to \pi - \theta$$
$$\phi \to \pi - \phi$$

rotation about y of π

also called Pauli-Y gate



Gate	Transformation on Bloch sphere (defined for single qubit)
X	π-rotation around the X axis, Z→−Z. Also referred to as a bit-flip.
Z	π-rotation around the Z axis, X→−X. Also referred to as a phase-flip.
H	maps X→Z, and Z→X. This gate is required to make superpositions.
S	maps X→Y. This gate extends H to make complex superpositions. (π/2 rotation around Z axis).
$S^\dagger$	inverse of S. maps X→−Y. (-π/2 rotation around Z axis).
T	π/4 rotation around Z axis.
$T^\dagger$	-π/4 rotation around Z axis.

Hadamard (H) gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- It maps

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longleftrightarrow |x_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow |x_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- for other states, it acts as a rotation about z of π, followed by a rotation about y of π/2

Phase gate

- Phase gates are defined $R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$

- when $\phi = \pi$ $R_\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is Pauli-Z gate

- when $\phi = \dfrac{\pi}{2}$ $R_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \sqrt{Z}$

  rotation about z of π/2 (called S in IBM Q)

- when $\phi = \dfrac{\pi}{4}$ $R_{\pi/2} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1+i \end{pmatrix}$

  (called T in IBM Q)

Square root of NOT gate

$$\sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

rotation about x of π/2 also called $\sqrt{\text{NOT}}$

Spin dynamics

- Schrodinger equation $$i\hbar\frac{d\psi}{dt} = H\psi = \frac{eg\hbar}{4m_e}\sigma\cdot\mathbf{B}\psi$$

- If B in z-direction $$i\hbar\frac{d\psi}{dt} = \frac{eg\hbar}{4m_e}\sigma_z\psi$$

- the spinor state $$\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$$

- for the energy eigenstate $$\psi(t) = e^{-i\omega t}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

eigenstate

- eigen equation $$\frac{eg}{4m_e}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

- eigenstates $$\omega = \pm \frac{eg}{4m_e} = \pm \omega_0$$ $$\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ $$\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- general solution

$$\psi(t) = ae^{-i\omega_0 t}\phi_+ + be^{i\omega_0 t}\phi_- = \begin{pmatrix} ae^{-i\omega_0 t} \\ be^{i\omega_0 t} \end{pmatrix}$$

spin precession

$$\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \qquad |u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\phi}{2}} \\ e^{i\frac{\phi}{2}} \end{pmatrix}$$

- Set initial state to be in x-direction

$$\phi = 0 \qquad \psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

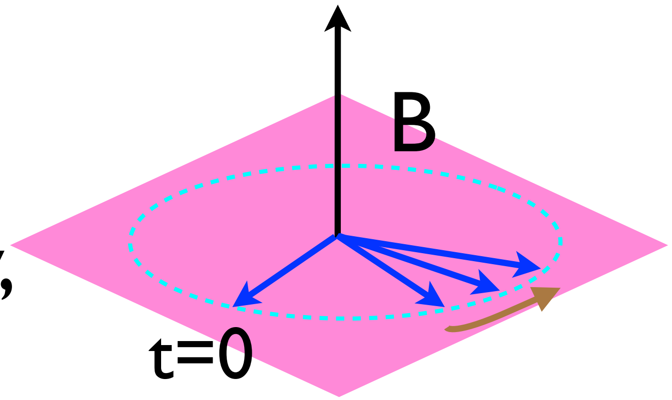
- for arbitrary time

$$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$$

- The expectation value

$$\langle S_x \rangle = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} e^{i\omega_0 t} & e^{-i\omega_0 t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{4}\left(e^{2i\omega_0 t} + e^{-2i\omega_0 t}\right) = \frac{\hbar \cos 2\omega_0 t}{2}$$

spin precession



- The spin precession frequency, called Larmor frequency

$$\Omega = 2\omega_0 = \frac{egB}{2m_e} = g\omega_c$$

- For B=1T, ω_c ~ 0.9 x 10^{11} rad/s

Paramagnetic resonance

- The magnetic field has a small oscillating part

$$\mathbf{B} = B_0\hat{z} + B_1\cos\omega t\hat{x}$$

- solve the Schrodinger equation

$$i\hbar\frac{d}{dt}\psi = \frac{eg\hbar}{4m_e}\begin{pmatrix} B_0 & B_1\cos\omega t \\ B_1\cos\omega t & -B_0 \end{pmatrix}\psi \qquad \psi = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$i\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{eg}{4m_e}\begin{pmatrix} B_0 & B_1\cos\omega t \\ B_1\cos\omega t & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

- When B_1=0

$$\psi_0 = \begin{pmatrix} a(0)e^{-i\omega_0 t} \\ b(0)e^{i\omega_0 t} \end{pmatrix}$$

Paramagnetic resonance

- When B_1 <> 0, the solution $\psi \approx \psi_0$
- Slowly varying functions A and B

$$a(t)e^{i\omega_0 t} = A(t)$$
$$b(t)e^{-i\omega_0 t} = B(t)$$

- Consider how A and B evolve with time

$$i\frac{dA(t)}{dt} = i\frac{da(t)}{dt}e^{i\omega_0 t} - \omega_0 a(t)e^{i\omega_0 t} = \omega_0 a(t)e^{i\omega_0 t} + \omega_1 b(t)\cos(\omega t)e^{i\omega_0 t} - \omega_0 A(t)$$

$$= \omega_1 b(t)\cos(\omega t)e^{i\omega_0 t} = \omega_1 B(t)\cos(\omega t)e^{2i\omega_0 t} = \frac{1}{2}\omega_1 B(t)\left(e^{2i\omega_0 t + i\omega t} + e^{2i\omega_0 t - i\omega t}\right)$$

$$i\frac{dB(t)}{dt} = \frac{1}{2}\omega_1 A(t)\left(e^{-2i\omega_0 t + i\omega t} + e^{-2i\omega_0 t - i\omega t}\right) \qquad \omega_1 = \frac{egB_1}{4m_e}$$

Rotating wave approximation

- When the driving frequency is close resonance that

$$\omega \approx 2\omega_0$$

- There are rapid oscillating and slow oscillating terms
- The rotating wave approximation states that only slow oscillating term is important

$$\left(e^{\pm 2i\omega_0 t + i\omega t} + e^{\pm 2i\omega_0 t - i\omega t}\right) \simeq e^{\pm\left(2i\omega_0 t - i\omega t\right)}$$

Rabi oscillation

- To solve the coupled equation

$$i\frac{dA(t)}{dt} \approx \frac{1}{2}\omega_1 B(t) e^{2i\omega_0 t - i\omega t} \qquad i\frac{dB(t)}{dt} \approx \frac{1}{2}\omega_1 A(t) e^{-2i\omega_0 t + i\omega t}$$

$$\frac{d^2A(t)}{dt^2} \approx -\frac{i}{2}\omega_1 e^{2i\omega_0 t - i\omega t}\frac{dB(t)}{dt} + \frac{1}{2}\omega_1(2\omega_0 - \omega)e^{2i\omega_0 t - i\omega t}B(t)$$

$$= \left(\frac{\omega_1}{2}\right)^2 A(t) + i(2\omega_0 - \omega)\frac{dA(t)}{dt}$$

- The solution is Rabi frequency

$$A(t) = A(0)e^{i\Omega t} \qquad -\Omega^2 = \left(\frac{\omega_1}{2}\right)^2 - (2\omega_0 - \omega)\Omega$$

$$\Omega = \left(\omega_0 - \frac{\omega}{2}\right) \pm \sqrt{\left(\omega_0 - \frac{\omega}{2}\right)^2 + \left(\frac{\omega_1}{2}\right)^2}$$

State evolution

- General solution $A(t) = A_+ e^{i\Omega_+ t} + A_- e^{i\Omega_- t}$

$$B(t) = e^{-2i\omega_0 t + i\omega t} \frac{2i}{\omega_1} \frac{dA(t)}{dt} = -\frac{2}{\omega_1} e^{-2i\omega_0 t + i\omega t} \left(A_+ \Omega_+ e^{i\Omega_+ t} + A_- \Omega_- e^{i\Omega_- t} \right)$$

$$= -\frac{2}{\omega_1} \left(A_+ \Omega_+ e^{-i\Omega_- t} + A_- \Omega_- e^{-i\Omega_+ t} \right)$$

- Suppose t=0 $\psi = \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- The coefficients

$$A(0) = a(0) = 1 \qquad A_+ + A_- = 1 \qquad A_+ = \frac{\Omega_-}{\Omega_- - \Omega_+}$$

$$B(0) = b(0) = 0 \qquad A_+\Omega_+ + A_-\Omega_- = 0 \qquad A_- = -\frac{\Omega_+}{\Omega_- - \Omega_+}$$

state evolution

- The probability to find the spin pointing in -z direction is

$$P_-(t) = |b(t)|^2 = |B(t)|^2 = \left(\frac{2}{\omega_1}\right)^2 \left|A_+\Omega_+ e^{-i\Omega_- t} + A_-\Omega_- e^{-i\Omega_+ t}\right|^2$$

$$= \left(\frac{2}{\omega_1}\right)^2 \left(\frac{\Omega_-\Omega_+}{\Omega_- - \Omega_+}\right)^2 \left|e^{-i\Omega_- t} - e^{-i\Omega_+ t}\right|^2$$

$$= 2\left(\frac{2}{\omega_1}\right)^2 \left(\frac{\Omega_-\Omega_+}{\Omega_- - \Omega_+}\right)^2 \left[1 - \cos(\Omega_- - \Omega_+)t\right]$$

$$= \frac{1}{2}\frac{\omega_1^2}{(2\omega_0 - \omega)^2 + \omega_1^2}\left[1 - \cos\sqrt{(2\omega_0 - \omega)^2 + \omega_1^2 t}\right]$$

$$\Omega_+\Omega_- = -\left(\frac{\omega_1}{2}\right)^2$$

$$\Omega_+ + \Omega_- = 2\omega_0 - \omega$$

$$\Omega_+ - \Omega_- = \sqrt{(2\omega_0 - \omega)^2 + \omega_1^2}$$

resonance condition

- when $\omega = 2\omega_0$ $\Omega = \pm\frac{\omega_1}{2}$

- The down-spin probability $P_-(t) = \frac{1}{2}(1 - \cos\omega_1 t)$

- For nuclear spin $\omega_1 = \frac{egB_1}{4m_n}$

Nuclear magnetic resonance

Particle	Spin	w_{Larmor}/B $s^{-1}T^{-1}$	n/B
Electron	1/2	1.7608×10^{11}	28.025 GHz/T
Proton	1/2	2.6753×10^{8}	42.5781 MHz/T
Deuteron	1	0.4107×10^{8}	6.5357 MHz/T
Neutron	1/2	1.8326×10^{8}	29.1667 MHz/T
^{23}Na	3/2	0.7076×10^{8}	11.2618 MHz/T
^{31}P	1/2	1.0829×10^{8}	17.2349 MHz/T
^{14}N	1	0.1935×10^{8}	3.08 MHz/T
^{13}C	1/2	0.6729×10^{8}	10.71 MHz/T
^{19}F	1/2	2.518×10^{8}	40.08 MHz/T

900MHz, B=21.1 T

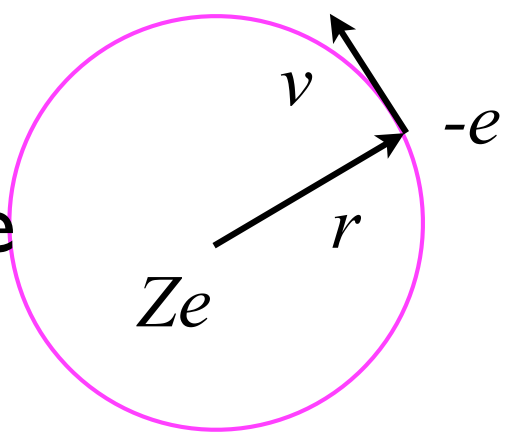
magnetic field in atoms

- electron spin may have interaction with internal magnetic field of an atom.
- at the moving frame, the nucleus may produce a magnetic field

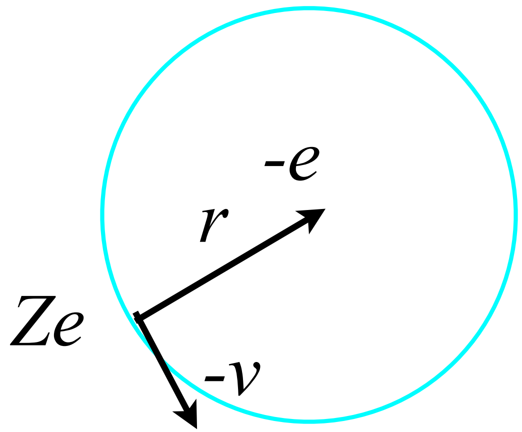
$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3} \qquad I d\vec{l} = -\frac{Ze\vec{v}}{2\pi r}dl$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}\times\vec{r}}{r^3} = -\frac{Ze\mu_0}{4\pi}\frac{\vec{v}\times\vec{r}}{r^3}$$

rest frame



moving frame



field transformation

- The B field is related to the Coulomb electric field

$$\vec{E} = \frac{Ze}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} \qquad \vec{B} = -\varepsilon_0\mu_0\vec{v}\times\vec{E} = -\frac{1}{c^2}\vec{v}\times\vec{E}$$

- This is similar to the transformation in special relativity

$$E'_\perp = \gamma\left(E_\perp + \vec{v}\times\vec{B}\right) \qquad B'_\perp = \gamma\left(B_\perp - \frac{1}{c^2}\vec{v}\times\vec{E}\right) \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

spin interaction

- The magnetic field produces an energy change to the electron

$$\Delta E = -\vec{\mu}_s \cdot \vec{B} = \frac{g_s \mu_b}{\hbar} \vec{S} \cdot \vec{B}$$

- The energy change transformation back to the rest frame would be reduced by half

$$\Delta E = \frac{g_s \mu_b}{2\hbar} \vec{S} \cdot \vec{B}$$

spin-orbit interaction

- To combine the two equations and note that

$$\vec{E} = -\frac{\vec{F}}{e} = \frac{1}{e}\frac{dV}{dr}\frac{\vec{r}}{r}$$

$$\vec{B} = -\frac{1}{ec^2 r}\frac{dV}{dr}\vec{v}\times\vec{r} = \frac{1}{emc^2 r}\frac{dV}{dr}\vec{L} \qquad \vec{L} = m\vec{r}\times\vec{v}$$

$$\Delta E = \frac{1}{emc^2 r}\frac{dV}{dr}\frac{g_s \mu_b}{2\hbar}\vec{S}\cdot\vec{L} = \frac{1}{2m^2 c^2 r}\frac{dV}{dr}\vec{S}\cdot\vec{L}$$

in solids

- In semiconductors, the crystal may has internal electric field E
- The E field produces a B field in the electron moving frame

$$\vec{B} = -\frac{1}{c^2}\vec{v} \times \vec{E}$$

- The B field produces energy change

$$\Delta E = -\frac{e\hbar}{4m}\vec{\sigma} \cdot \left(\vec{v} \times \vec{E}\right)$$

$$= -\frac{e\hbar^2}{4m^2}\vec{\sigma} \cdot \left(\vec{k} \times \vec{E}\right)$$

$$\vec{v} = \frac{\hbar\vec{k}}{m}$$

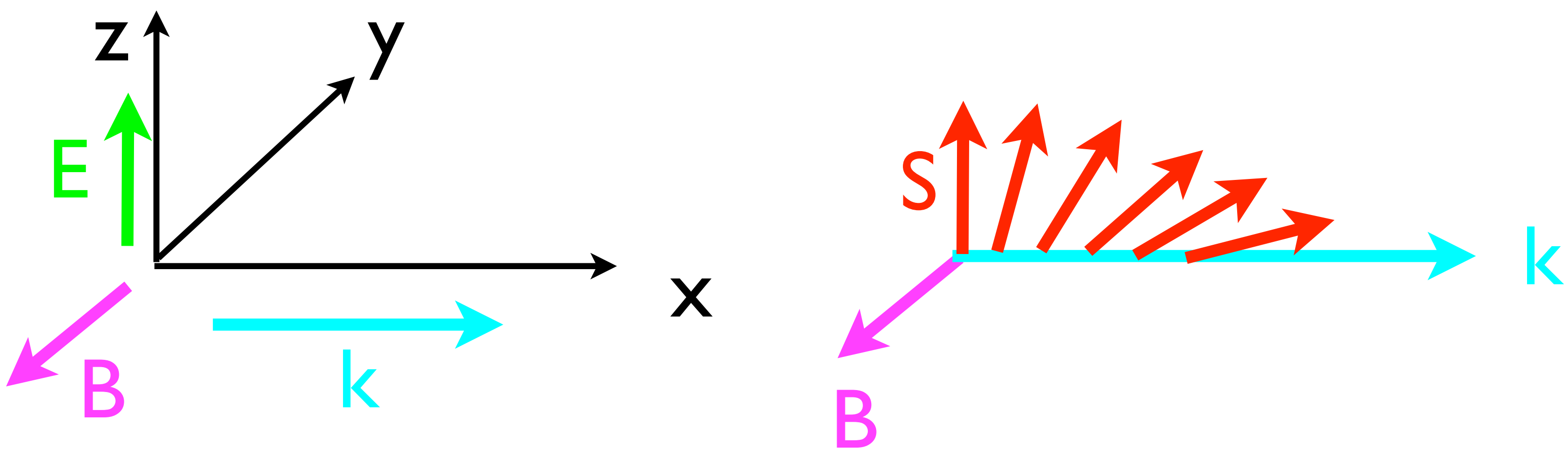
Rashba effect

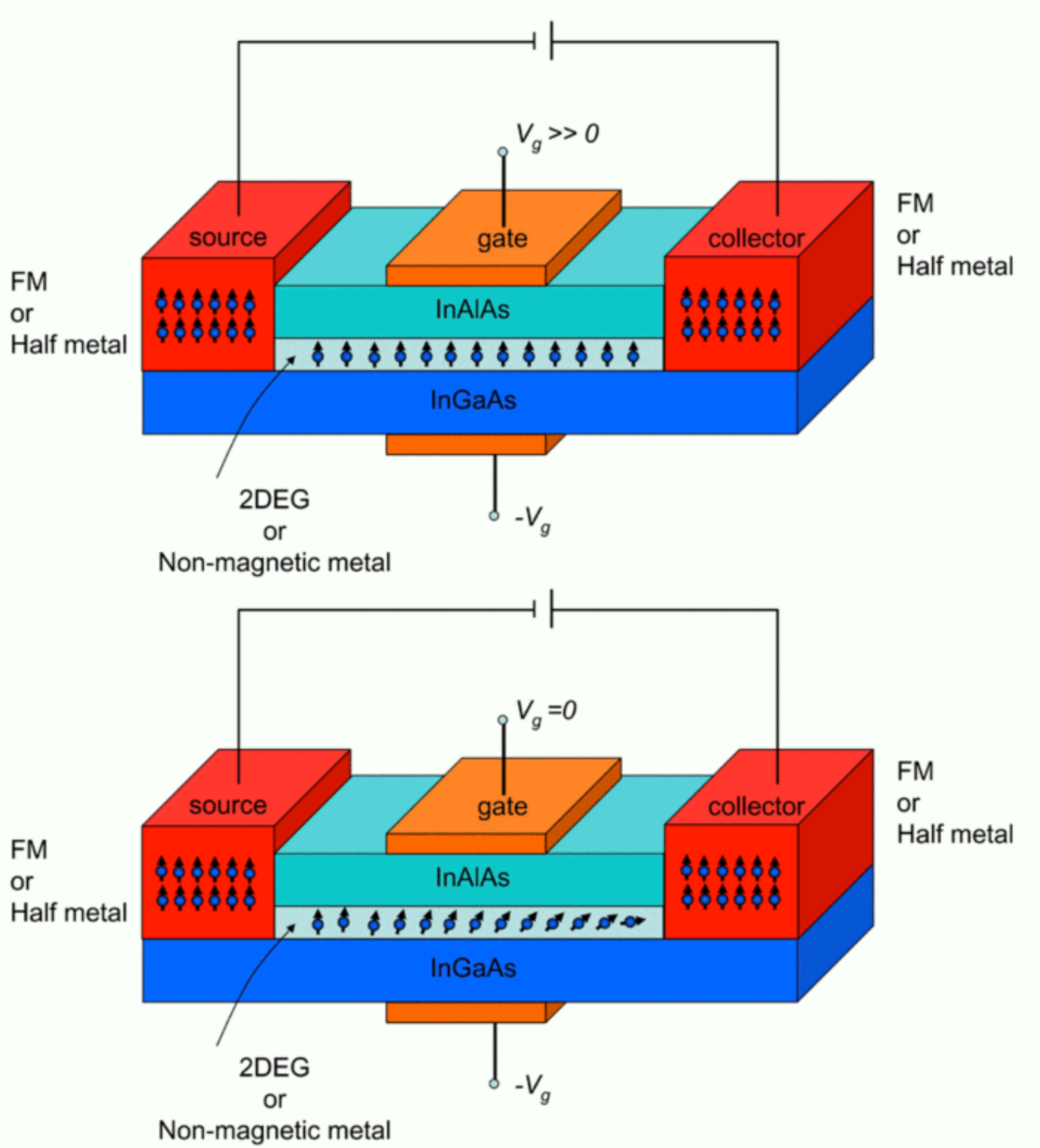
- The Rashba effect states that

$$\vec{E} = E_0\hat{z}$$

$$\Delta E = -\frac{e\hbar^2 E_0}{4m^2}\left(\vec{\sigma}\times\vec{k}\right)\cdot\hat{z}$$

- The spin would precess when moving forward





V_g >> 0
source
gate
collector
FM
or
Half metal
FM
or
Half metal
InAlAs
InGaAs
2DEG
or
Non-magnetic metal
-V_g
V_g =0
source
gate
collector
FM
or
Half metal
FM
or
Half metal
InAlAs
InGaAs
2DEG
or
Non-magnetic metal
-V_g

Addition of two spins

- The 2 spin system
- electron 1 $\quad \left[S_{1x}, S_{1y}\right] = i\hbar S_{1z}$
- electron 2 $\quad \left[S_{2x}, S_{2y}\right] = i\hbar S_{2z}$

$$\left[S_{1i}, S_{2j}\right] = 0 \qquad \text{for all } i, j$$

Total spin

- Total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$
- commutation relation

$$
\begin{aligned}
\left[S_x, S_y\right] &= \left[S_{1x} + S_{2x}, S_{1y} + S_{2y}\right] \\
&= \left[S_{1x}, S_{1y}\right] + \left[S_{2x}, S_{2y}\right] \\
&= i\hbar S_{1z} + i\hbar S_{2z} \\
&= i\hbar S_z
\end{aligned}
$$

- Therefor it is easy to find total spin S satisfies the commutation relation of an angular momentum

Eigenvalues

- Consider the states using single spinors
- electron 1 $\chi_{\pm}^{(1)}$

$$S_1^2 \chi_{\pm}^{(1)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_{\pm}^{(1)}$$

$$S_{1z} \chi_{\pm}^{(1)} = \pm\frac{1}{2}\hbar \chi_{\pm}^{(1)}$$

- electron 2 $\chi_{\pm}^{(2)}$

$$S_2^2 \chi_{\pm}^{(2)} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \chi_{\pm}^{(2)}$$

$$S_{2z} \chi_{\pm}^{(2)} = \pm\frac{1}{2}\hbar \chi_{\pm}^{(2)}$$

product states

- The possible states are (product states)

$$\chi_+^{(1)}\chi_+^{(2)} \qquad \chi_+^{(1)}\chi_-^{(2)} \qquad \chi_-^{(1)}\chi_+^{(2)} \qquad \chi_-^{(1)}\chi_-^{(2)}$$

- calculate the eigenvalues

$$\begin{aligned} S_z\chi_+^{(1)}\chi_+^{(2)} &= \left(S_{1z} + S_{2z}\right)\chi_+^{(1)}\chi_+^{(2)} \\ &= \left(S_{1z}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}\left(S_{2z}\chi_+^{(2)}\right) \\ &= \hbar\chi_+^{(1)}\chi_+^{(2)} \end{aligned}$$

$$S_z\chi_+^{(1)}\chi_-^{(2)} = S_z\chi_-^{(1)}\chi_+^{(2)} = 0 \qquad S_z\chi_-^{(1)}\chi_-^{(2)} = -\hbar\chi_-^{(1)}\chi_-^{(2)}$$

- Two m=0 states

spin triplet and singlet

- Spin triplet S=1, m=1, 0, -1
- Spin singlet S=0, m=0
- May check using lowering operator $S_- = S_{1-} + S_{2-}$

$$S_{1-}\chi_+^{(1)} = \hbar\chi_-^{(1)}$$
$$S_{2-}\chi_+^{(2)} = \hbar\chi_-^{(2)}$$

$$S_-\chi_+^{(1)}\chi_+^{(2)} = \left(S_{1-}\chi_+^{(1)}\right)\chi_+^{(2)} + \chi_+^{(1)}\left(S_{2-}\chi_+^{(2)}\right)$$
$$= \hbar\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$

- S=1, m=0 state $X_+ = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$

spin triplet and singlet

- One may check the result again

$$S_- \frac{\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}}{\sqrt{2}} = \left(S_{1-} + S_{2-}\right)\frac{\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\left(S_{1-}\chi_+^{(1)}\right)\chi_-^{(2)} + \frac{1}{\sqrt{2}}\chi_-^{(1)}\left(S_{2-}\chi_+^{(2)}\right)$$
$$= \sqrt{2}\hbar\chi_-^{(1)}\chi_-^{(2)}$$

- The remaining state m=0

$$X_- = \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} - \chi_+^{(1)}\chi_-^{(2)}\right)$$

S^2

- check the S^2 value

$$\mathbf{S}^2 = \left(\mathbf{S}_1 + \mathbf{S}_2\right)^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$$
$$= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z}$$
$$= \mathbf{S}_1^2 + \mathbf{S}_2^2 + S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}$$

$$\mathbf{S}_1^2 X_+ = \frac{1}{\sqrt{2}}\mathbf{S}_1^2\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$
$$= \frac{3}{4}\hbar^2 \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right) = \frac{3}{4}\hbar^2 X_+$$

$$\mathbf{S}_1^2 X_- = \frac{3}{4}\hbar^2 X_-$$

$$\mathbf{S}_2^2 X_- = \frac{3}{4}\hbar^2 X_-$$

$$\mathbf{S}_2^2 X_+ = \frac{3}{4}\hbar^2 X_+$$

$$S_{1z}S_{2z}X_+ = \frac{1}{\sqrt{2}}S_{1z}S_{2z}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right)$$
$$= \frac{1}{\sqrt{2}}S_{1z}\chi_-^{(1)}S_{2z}\chi_+^{(2)} + \frac{1}{\sqrt{2}}S_{1z}\chi_+^{(1)}S_{2z}\chi_-^{(2)}$$
$$= -\frac{1}{4}\hbar^2 \frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)} + \chi_+^{(1)}\chi_-^{(2)}\right) = -\frac{1}{4}\hbar^2 X_+$$

$$S_{1z}S_{2z}X_- = -\frac{1}{4}\hbar^2 X_-$$

S^2

$$\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)X_+ = \frac{1}{\sqrt{2}}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)\left(\chi_-^{(1)}\chi_+^{(2)}+\chi_+^{(1)}\chi_-^{(2)}\right)$$

$$= \frac{1}{\sqrt{2}}\left(S_{1+}\chi_-^{(1)}\right)\left(S_{2-}\chi_+^{(2)}\right)+\frac{1}{\sqrt{2}}\left(S_{1-}\chi_+^{(1)}\right)\left(S_{2+}\chi_-^{(2)}\right)$$

$$= \frac{1}{\sqrt{2}}\hbar^2\left(\chi_+^{(1)}\chi_-^{(2)}+\chi_-^{(1)}\chi_+^{(2)}\right)=\hbar^2 X_+$$

$$\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)X_- = \frac{1}{\sqrt{2}}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)\left(\chi_-^{(1)}\chi_+^{(2)}-\chi_+^{(1)}\chi_-^{(2)}\right)$$

$$= \frac{1}{\sqrt{2}}\left(S_{1+}\chi_-^{(1)}\right)\left(S_{2-}\chi_+^{(2)}\right)-\frac{1}{\sqrt{2}}\left(S_{1-}\chi_+^{(1)}\right)\left(S_{2+}\chi_-^{(2)}\right)$$

$$= -\frac{1}{\sqrt{2}}\hbar^2\left(\chi_+^{(1)}\chi_-^{(2)}-\chi_-^{(1)}\chi_+^{(2)}\right)=-\hbar^2 X_-$$

S^2

- For X_+ , $S=1$

$$
\begin{aligned}
\mathbf{S}^2 X_+ &= \mathbf{S}_1^2 X_+ + \mathbf{S}_2^2 X_+ + S_{1+} S_{2-} X_+ + S_{1-} S_{2+} X_+ + 2 S_{1z} S_{2z} X_+ \\
&= \frac{3}{4}\hbar^2 X_+ + \frac{3}{4}\hbar^2 X_+ + \hbar^2 X_+ - \frac{1}{2}\hbar^2 X_+ \\
&= 2\hbar^2 X_+ = S(S+1)\hbar^2 X_+
\end{aligned}
$$

- For X_- , $S=0$

$$
\begin{aligned}
\mathbf{S}^2 X_- &= \mathbf{S}_1^2 X_- + \mathbf{S}_2^2 X_- + S_{1+} S_{2-} X_- + S_{1-} S_{2+} X_- + 2 S_{1z} S_{2z} X_- \\
&= \frac{3}{4}\hbar^2 X_- + \frac{3}{4}\hbar^2 X_- - \hbar^2 X_- - \frac{1}{2}\hbar^2 X_- \\
&= 0
\end{aligned}
$$

representation

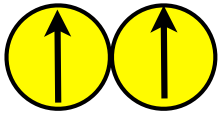
- product states
- total spin state

Spin triplet

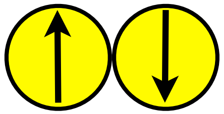
Spin singlet

S=1

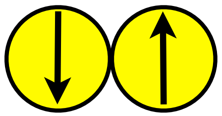
S=0



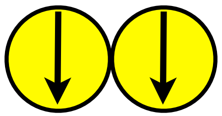
$\chi_+^{(1)}\chi_+^{(2)}$



$\chi_+^{(1)}\chi_-^{(2)}$



$\chi_-^{(1)}\chi_+^{(2)}$



$\chi_-^{(1)}\chi_-^{(2)}$

Spin triplet (S=1):

$\chi_+^{(1)}\chi_+^{(2)}$

$$\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)}+\chi_+^{(1)}\chi_-^{(2)}\right)$$

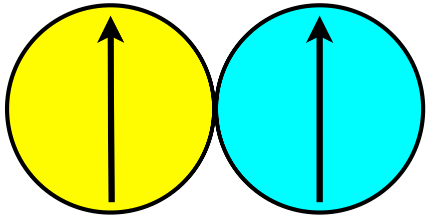
$\chi_-^{(1)}\chi_-^{(2)}$

Spin singlet (S=0):

$$\frac{1}{\sqrt{2}}\left(\chi_-^{(1)}\chi_+^{(2)}-\chi_+^{(1)}\chi_-^{(2)}\right)$$

spin-dependent potential

- In many physical systems, two particle interaction is spin-dependent
- the duetron hamiltonian



$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V_1(r) + \frac{1}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2 V_2(r)$$

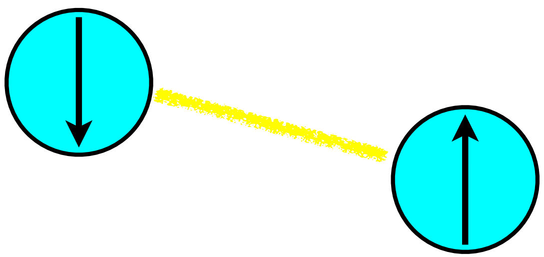
$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right) = \frac{1}{2}\mathbf{S}^2 - \frac{3}{4}\hbar^2$$

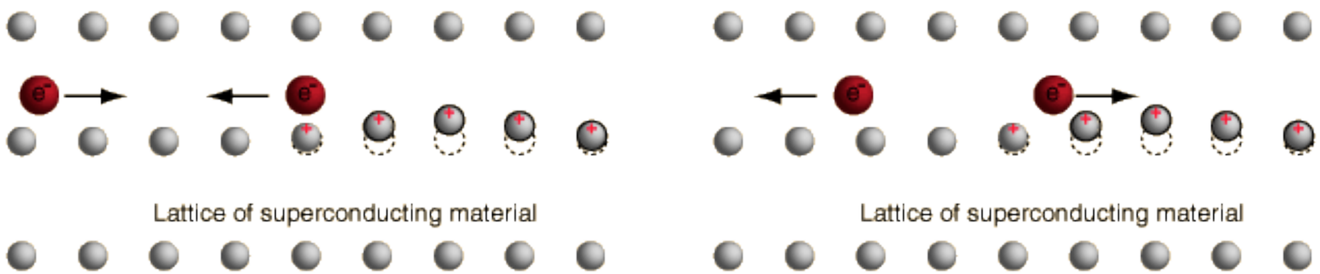
- S^2 is a good quantum number, but S_z is not
- for triplet $V(r) = V_1(r) + \left(1 - \frac{3}{4}\right)V_2(r) = V_1(r) + \frac{1}{4}V_2(r)$
- for singlet $V(r) = V_1(r) + \left(0 - \frac{3}{4}\right)V_2(r) = V_1(r) - \frac{3}{4}V_2(r)$

spin-dependent potential

- for deutron, one observes a bound S=1 state and an unbound S=0 state

- for BCS paring, bound state S=0

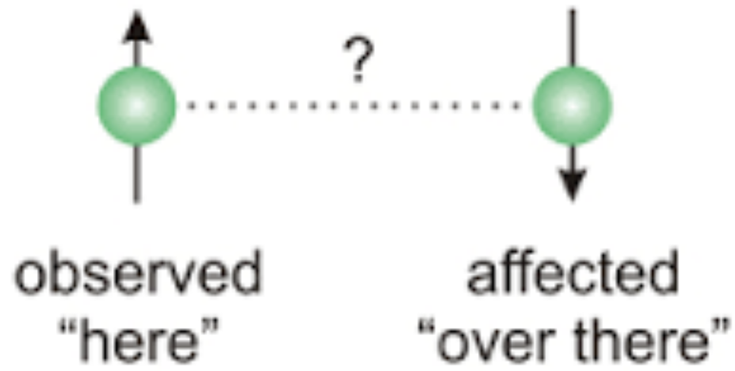




http://hyperphysics.phy-astr.gsu.edu/hbase/Solids/coop.html

spin singlet and entanglement

- In the spin singlet, quantum states are entangled
- First we do S_x measurement on electron 1, we have 50% to get `+' and 50% to get `-'
- then we do S_x measurement on electron 2, the result is 100% opposite to the result of electron 1.



How does it work?

- entangled state $$\psi = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2\right)$$

- the measurement of S_{x1} project the state to an eigenstate of S_{x1}

$$\left|S_x = +\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_{x1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_1(+) = \left|S_x = +\right\rangle\left\langle S_x = +\right|$$

- The project operator

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

measurement

- Projection result

$$P_1(+)\psi = \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2$$

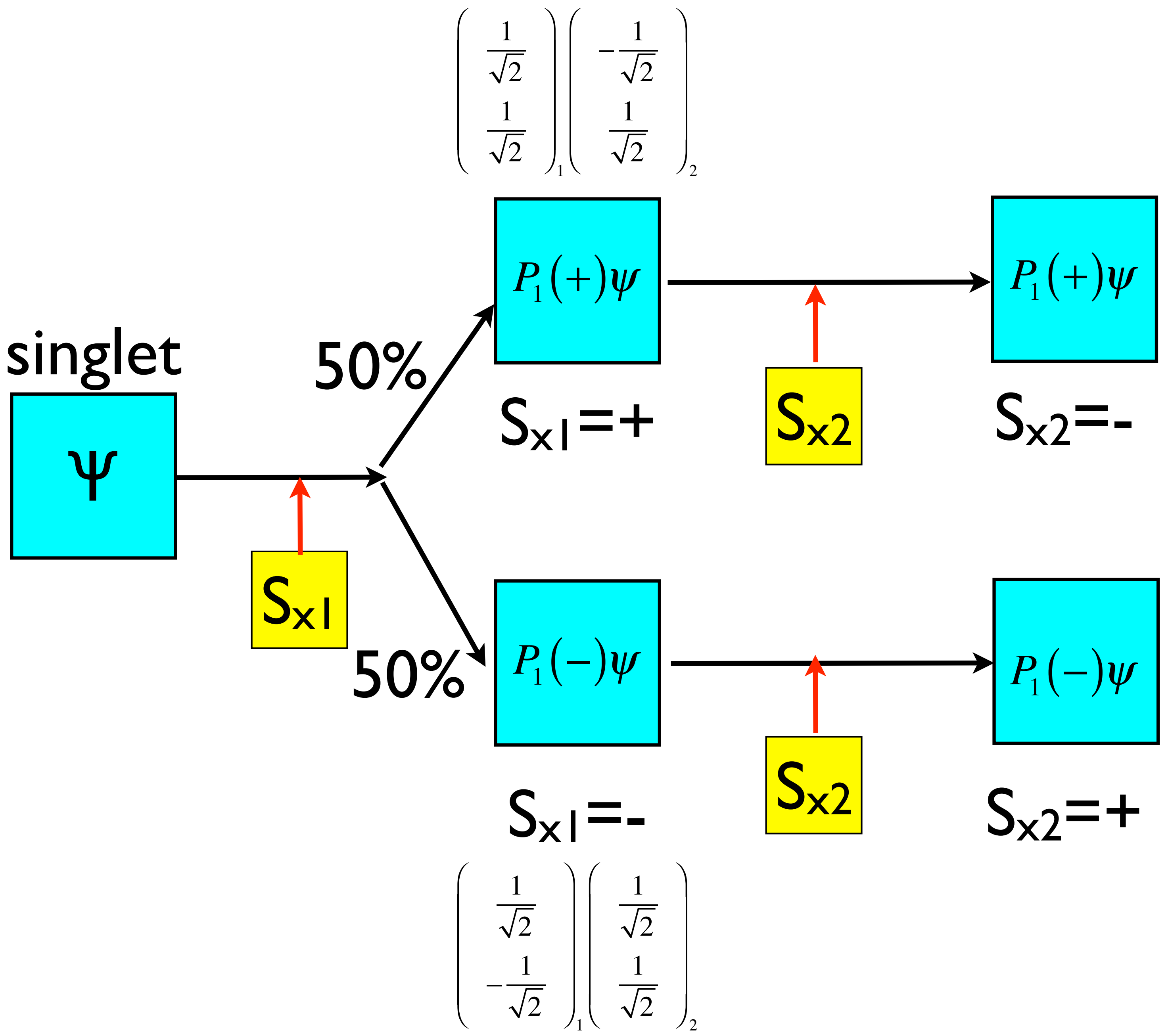
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_1 \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}_2$$

$$= \psi'$$

- The following measurement on S_{x2} will only give `-' result

$$S_{x2}\psi' = S_{x2}P_1(+)\psi = -\frac{\hbar}{2}\psi'$$


singlet
Ψ
Sx1
50%
50%
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_1 \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_2$
$P_1(+)\psi$
Sx1=+
Sx2
$P_1(+)\psi$
Sx2=-
$P_1(-)\psi$
Sx1=-
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_2$
Sx2
$P_1(-)\psi$
Sx2=+

- Einstein’s comment: “spukhafte Fernwirkung” or "spooky action at a distance

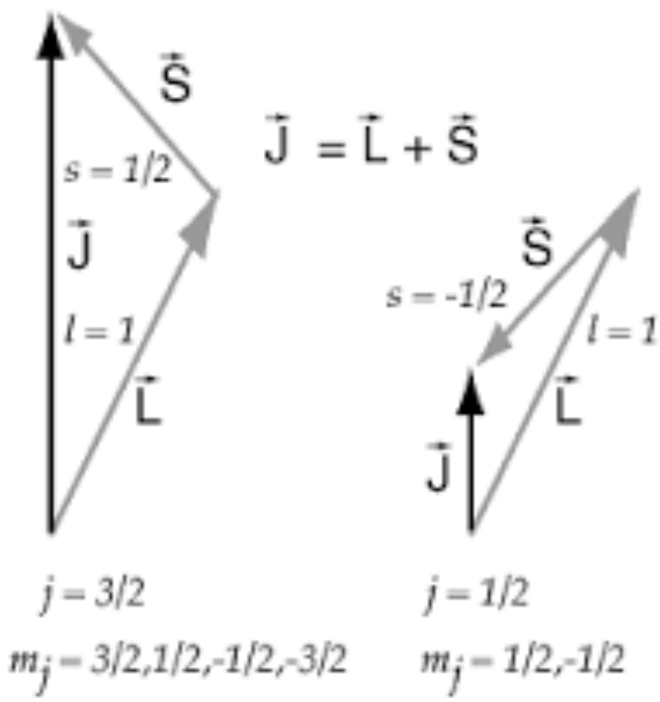
Addition of L and S

- total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$
- product state $Y_{lm}\chi_{\pm}$
- eigenstate $\mathbf{J}^2\psi_{j,m_j} = \hbar^2 j(j+1)\psi_{j,m_j}$

  $J_z\psi_{j,m_j} = \hbar m_j \psi_{j,m_j}$

- eigenvalue $j = l \pm \frac{1}{2}$

$$m_j = -j, -j+1 \cdots, j-1, j$$



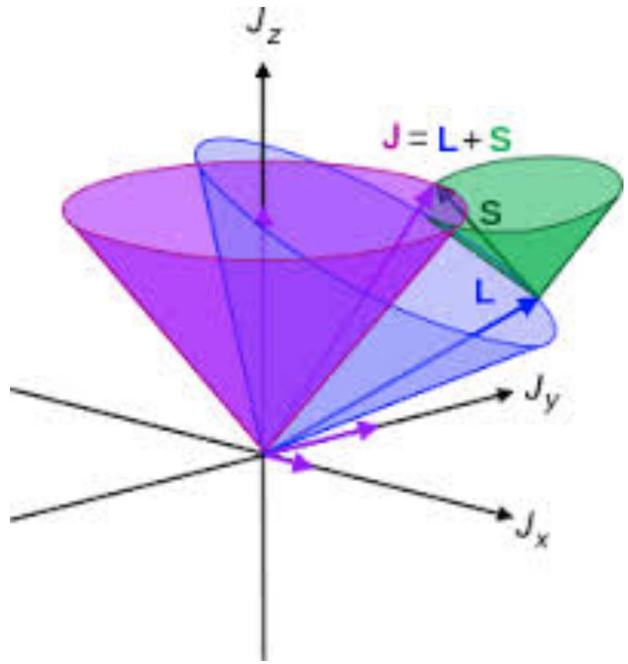
Addition of L and S

- case 1 $j = l + \frac{1}{2}$ $m_j = m + \frac{1}{2}$

$$\psi_{j,m_j} = \sqrt{\frac{l+m+1}{2l+1}} Y_{lm}\chi_+ + \sqrt{\frac{l-m}{2l+1}} Y_{lm+1}\chi_-$$

- case 2 $j = l - \frac{1}{2}$ $m_j = m + \frac{1}{2}$

$$\psi_{j,m_j} = \sqrt{\frac{l-m}{2l+1}} Y_{lm}\chi_+ + \sqrt{\frac{l+m+1}{2l+1}} Y_{lm+1}\chi_-$$



Addition of angular momenta

$$\mathbf{J} = \mathbf{L}_1 + \mathbf{L}_2$$

- possible total angular momentum

$$j = l_1 + l_2, l_1 + l_2 - 1, \cdots |l_1 - l_2|$$

- possible z-component

$$m_j = -j, -j + 1 \cdots, j - 1, j$$

Fine structure

- Consider the total angular momentum

$$\vec{J} = \vec{L} + \vec{S} \qquad J^2 = \left(\vec{L} + \vec{S}\right)^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$$

- For angular momentum eigenstates, we have

$$\vec{L}\cdot\vec{S} = \frac{\hbar^2}{2}\left[\, j(j+1) - l(l+1) - s(s+1)\right]$$

- The spin-orbital energy is

$$\Delta E = \frac{\hbar^2}{4m^2c^2}\left\langle \frac{1}{r}\frac{dV}{dr} \right\rangle \left[j(j+1) - l(l+1) - s(s+1) \right]$$

$$\begin{aligned} & j(j+1) - l(l+1) - s(s+1) \\ &= \left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) - l(l+1) - s(s+1) \\ &= l \end{aligned}$$

$$\begin{aligned} & j(j+1) - l(l+1) - s(s+1) \\ &= \left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right) - l(l+1) - s(s+1) \\ &= -l-1 \end{aligned}$$

- The electron energy is on the order of

$$\begin{aligned} E_0 &\sim -\frac{e^2}{8\pi\varepsilon_0}\left\langle \frac{1}{r} \right\rangle = -\frac{e^2}{8\pi\varepsilon_0}\frac{1}{a_0} \\ &= \frac{e^2}{8\pi\varepsilon_0}\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right) = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \end{aligned}$$

- The spin-orbit energy is on the order of

$$\Delta E \sim \frac{\hbar^2}{4m^2c^2}\frac{e^2}{4\pi\varepsilon_0}\left\langle\frac{1}{r^3}\right\rangle = \frac{\hbar^2}{4m^2c^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{a_0^3}$$

$$= \frac{\hbar^2}{4m^2c^2}\frac{e^2}{4\pi\varepsilon_0}\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right)^3 = \frac{m}{4c^2\hbar^4}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^4$$

$$= E_0\frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2 = \frac{1}{2}E_0\alpha^2$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \sim \frac{1}{137}$$ called fine structure constant