

Quantum optics

Two level system

- An two-level atom in an external time-varying electric field

- Atom Hamiltonian
$$H_0 = \begin{pmatrix} \hbar\omega_a & 0 \\ 0 & \hbar\omega_b \end{pmatrix}$$

- Perturbation: E-field in x direction
$$H_1 = eE_0 x \cos \omega t$$

- Matrix element
$$\langle a | H_1 | a \rangle = eE_0 \langle a | x | a \rangle \cos \omega t$$

- The parity symmetry
$$\langle a | x | a \rangle = \langle b | x | b \rangle = 0$$

Interaction picture

- The perturbation can be written as

$$H_1 = \begin{pmatrix} 0 & W \cos \omega t \\ W^* \cos \omega t & 0 \end{pmatrix} \quad W = eE_0 \langle a | x | b \rangle$$

- The interaction picture states that

$$|\psi_I(t)\rangle = e^{\frac{i}{\hbar} H_0 t} |\psi(t)\rangle$$

- Schrodinger equation

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} (H_0 + H_1) |\psi(t)\rangle$$

Interaction picture

- Schrodinger equation

$$\begin{aligned}\frac{d}{dt}|\psi_I(t)\rangle &= \frac{d}{dt}\left(e^{\frac{i}{\hbar}H_0t}|\psi(t)\rangle\right) \\ &= \frac{i}{\hbar}H_0e^{\frac{i}{\hbar}H_0t}|\psi(t)\rangle + e^{\frac{i}{\hbar}H_0t}\frac{d}{dt}|\psi(t)\rangle \\ &= \frac{i}{\hbar}H_0e^{\frac{i}{\hbar}H_0t}|\psi(t)\rangle - \frac{i}{\hbar}e^{\frac{i}{\hbar}H_0t}(H_0 + H_1)|\psi(t)\rangle \\ &= -\frac{i}{\hbar}e^{\frac{i}{\hbar}H_0t}H_1|\psi(t)\rangle = -\frac{i}{\hbar}e^{\frac{i}{\hbar}H_0t}H_1e^{-\frac{i}{\hbar}H_0t}e^{\frac{i}{\hbar}H_0t}|\psi(t)\rangle \\ &= -\frac{i}{\hbar}V(t)e^{\frac{i}{\hbar}H_0t}|\psi_I(t)\rangle\end{aligned}$$

$$\frac{d}{dt}|\psi_I(t)\rangle = -\frac{i}{\hbar}V(t)|\psi_I(t)\rangle \qquad V(t) = e^{\frac{i}{\hbar}H_0t}H_1e^{-\frac{i}{\hbar}H_0t}$$

Time dependence

- To calculate

$$e^{\frac{i}{\hbar}H_0t} = \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \quad (H_0)^n = \begin{pmatrix} (\hbar\omega_a)^n & 0 \\ 0 & (\hbar\omega_b)^n \end{pmatrix}$$

$$\begin{aligned} V(t) &= \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \begin{pmatrix} 0 & W \cos \omega t \\ W^* \cos \omega t & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_a t} & 0 \\ 0 & e^{-i\omega_b t} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega_a t} & 0 \\ 0 & e^{i\omega_b t} \end{pmatrix} \begin{pmatrix} 0 & We^{-i\omega_b t} \cos \omega t \\ W^* e^{-i\omega_a t} \cos \omega t & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & We^{i(\omega_a - \omega_b)t} \cos \omega t \\ W^* e^{-i(\omega_a - \omega_b)t} \cos \omega t & 0 \end{pmatrix} \end{aligned}$$

$$e^{i(\omega_a - \omega_b)t} \cos \omega t = \frac{1}{2} e^{i(\omega_a - \omega_b - \omega)t} + \frac{1}{2} e^{i(\omega_a - \omega_b + \omega)t}$$

Close to resonance

- When the driving frequency is close to the level spacing

$$\omega_a - \omega_b - \omega = \Delta \ll \omega$$

$$e^{i(\omega_a - \omega_b)t} \cos \omega t = \frac{1}{2} e^{i(\omega_a - \omega_b - \omega)t} + \frac{1}{2} e^{i(\omega_a - \omega_b + \omega)t}$$

slow oscillating Fast oscillating

- Rotational wave approximation: the fast oscillating term can be dropped

$$V(t) = \begin{pmatrix} 0 & \frac{W}{2} e^{i\Delta t} \\ \frac{W^*}{2} e^{-i\Delta t} & 0 \end{pmatrix}$$

Differential equation

- Parameterize the wavefunction

$$|\psi_I(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

- The equations

$$\frac{da(t)}{dt} = -\frac{i}{\hbar} \frac{W}{2} e^{i\Delta t} b(t)$$
$$\frac{db(t)}{dt} = -\frac{i}{\hbar} \frac{W^*}{2} e^{-i\Delta t} a(t)$$

- Single equation

$$\frac{d^2 a(t)}{dt^2} - i\Delta \frac{da(t)}{dt} + \frac{|W|^2}{4\hbar^2} a(t) = 0$$

Rabi frequency

- The solution is assumed $a(t) = e^{i\Omega t}$

$$\Omega^2 - \Delta\Omega - \frac{|W|^2}{4\hbar^2} = 0$$

$$\Omega = \frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + \frac{|W|^2}{4\hbar^2}}$$

- The general solution

$$a(t) = a_1 e^{i\Omega_1 t} + a_2 e^{i\Omega_2 t}$$

$$b(t) = \frac{2\hbar}{|W|} \left(\Omega_1 a_1 e^{-i\Omega_2 t} + \Omega_2 a_2 e^{-i\Omega_1 t} \right)$$

Initial state

- If the atom is initially at the upper state

$$a(0) = a_1 + a_2 = 1$$

$$b(0) = \frac{2\hbar}{|W|} (\Omega_1 a_1 + \Omega_2 a_2) = 0$$

$$a_1 = \frac{\Omega_2}{\Omega_2 - \Omega_1} \qquad a_2 = \frac{\Omega_1}{\Omega_1 - \Omega_2}$$

- The probability amplitude in the lower state

$$b(t) = \frac{2\hbar}{|W|} \frac{\Omega_1 \Omega_2}{\Omega_1 - \Omega_2} (e^{-i\Omega_2 t} - e^{-i\Omega_1 t})$$

Probability

- Calculate the probability

$$\begin{aligned} P(t) &= |b(t)|^2 = \frac{4\hbar^2}{|W|^2} \left(\frac{\Omega_1 \Omega_2}{\Omega_1 - \Omega_2} \right)^2 \left| e^{-i\Omega_2 t} - e^{-i\Omega_1 t} \right|^2 \\ &= \frac{4\hbar^2}{|W|^2} \left(\frac{\Omega_1 \Omega_2}{\Omega_1 - \Omega_2} \right)^2 \sin^2 \frac{(\Omega_1 - \Omega_2)t}{2} \\ &= \frac{|W|^2}{|W|^2 + \hbar^2 \Delta^2} \sin^2 \sqrt{\frac{\Delta^2}{4} + \frac{|W|^2}{4\hbar^2}} t \end{aligned}$$

Quantum treatment of E-field

- To write E-field as an operator
- Basis: the direct product of atom and photon state

$$H_0 = \begin{pmatrix} \hbar\omega_a + N\hbar\omega & 0 \\ 0 & \hbar\omega_b + (N+1)\hbar\omega \end{pmatrix} \begin{array}{l} |a\rangle \otimes |N\rangle \\ |b\rangle \otimes |N+1\rangle \end{array}$$

- Interaction

$$W = e \langle N | E_0 | N+1 \rangle \langle a | x | b \rangle = \sqrt{N+1} W_0 \quad E_0 \rightarrow (a + a^+)$$

- Rabi model

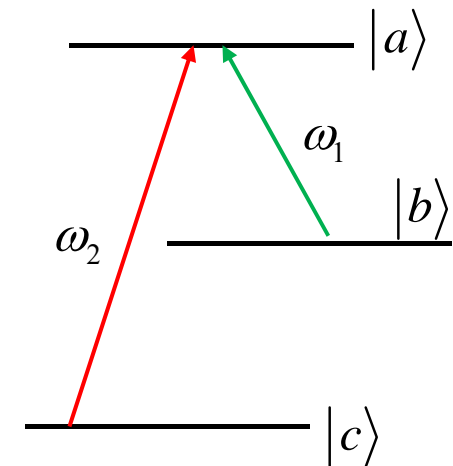
$$H = \hbar\omega_a \sigma_z + \hbar\omega a^+ a + g (a + a^+) \sigma_x$$

3 level system

- Atom Hamiltonian $H_0 = \begin{pmatrix} \hbar\omega_a & 0 & 0 \\ 0 & \hbar\omega_b & 0 \\ 0 & 0 & \hbar\omega_c \end{pmatrix}$

- Two-tone: $H_1 = eE_1x \cos \omega_1 t + eE_2x \cos \omega_2 t$

- Driving perturbation



$$H_1 = \begin{pmatrix} 0 & W_{1ab} \cos \omega_1 t + W_{2ab} \cos \omega_2 t & W_{1ac} \cos \omega_1 t + W_{2ac} \cos \omega_2 t \\ W_{1ab}^* \cos \omega_1 t + W_{2ab}^* \cos \omega_2 t & 0 & 0 \\ W_{1ac}^* \cos \omega_1 t + W_{2ac}^* \cos \omega_2 t & 0 & 0 \end{pmatrix}$$

Interaction picture and RWA

- In the interaction picture

$$V = \begin{pmatrix} 0 & X & Y \\ X^* & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix}$$

$$X = (W_{1ab} \cos \omega_1 t + W_{2ab} \cos \omega_2 t) e^{i(\omega_a - \omega_b)t}$$

$$Y = (W_{1ac} \cos \omega_1 t + W_{2ac} \cos \omega_2 t) e^{i(\omega_a - \omega_c)t}$$

- Detuning

$$\delta_1 = \omega_a - \omega_b - \omega_1$$

$$\delta_2 = \omega_a - \omega_c - \omega_2$$

$$X = \frac{W_{1ab}}{2} e^{i\delta_1 t}$$

$$Y = \frac{W_{1ac}}{2} e^{i\delta_2 t}$$

solution

- state $|\psi_I(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$

$$\frac{da(t)}{dt} = -\frac{i}{\hbar} \frac{W_{1ab}}{2} e^{i\delta_1 t} b(t) - \frac{i}{\hbar} \frac{W_{2ac}}{2} e^{i\delta_2 t} c(t)$$

$$\frac{db(t)}{dt} = -\frac{i}{\hbar} \frac{W_{1ab}}{2} e^{-i\delta_1 t} a(t)$$

$$\frac{dc(t)}{dt} = -\frac{i}{\hbar} \frac{W_{2ac}}{2} e^{-i\delta_2 t} a(t)$$

solution

• state $|\psi_I(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$

$$\frac{da(t)}{dt} = -\frac{i}{\hbar} \frac{W_{1ab}}{2} e^{i\delta_1 t} b(t) - \frac{i}{\hbar} \frac{W_{2ac}}{2} e^{i\delta_2 t} c(t)$$

$$\frac{db(t)}{dt} = -\frac{i}{\hbar} \frac{W_{1ab}^*}{2} e^{-i\delta_1 t} a(t)$$

$$\frac{dc(t)}{dt} = -\frac{i}{\hbar} \frac{W_{2ac}^*}{2} e^{-i\delta_2 t} a(t)$$

$$a(t) = A e^{i\Omega t}$$

$$b(t) = B e^{i\Omega t} e^{-i\delta_1 t}$$

$$c(t) = C e^{i\Omega t} e^{-i\delta_2 t}$$

$$-\hbar \begin{pmatrix} \Omega & 0 & 0 \\ 0 & \Omega - \delta_1 & 0 \\ 0 & 0 & \Omega - \delta_2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & 0 & 0 \\ \frac{W_{2ac}^*}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Eigen modes

$$\begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & -\hbar\delta_1 & 0 \\ \frac{W_{2ac}^*}{2} & 0 & -\hbar\delta_2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = -\hbar\Omega \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Omega(\Omega - \delta_1)(\Omega - \delta_2) - \frac{|W_{2ac}|^2}{4\hbar^2}(\Omega - \delta_1) - \frac{|W_{2ab}|^2}{4\hbar^2}(\Omega - \delta_2) = 0$$

Perfect tuning

- In resonance $\delta_1 = \delta_2 = 0$

$$\Omega^3 - \left(\frac{|W_{2ac}|^2}{4\hbar^2} + \frac{|W_{2ab}|^2}{4\hbar^2} \right) \Omega = 0$$

- Three modes

$$\Omega = 0, \pm \frac{1}{2\hbar} \sqrt{|W_{2ac}|^2 + |W_{2ab}|^2}$$

Eigen modes for perfect tuning

- zero frequency mode

$$|\psi_I(t)\rangle = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \frac{1}{\sqrt{|W_{1ab}|^2 + |W_{2ac}|^2}} \begin{pmatrix} 0 \\ W_{2ac} \\ -W_{1ab} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & 0 & 0 \\ \frac{W_{2ac}^*}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

- Non-zero mode

$$|\psi_I(t)\rangle = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\Omega t} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{W_{1ab}^*}{2\hbar\Omega} \\ \frac{W_{2ac}^*}{2\hbar\Omega} \end{pmatrix} e^{i\Omega t}$$

$$\begin{pmatrix} \Omega & \frac{W_{1ab}}{2} & \frac{W_{2ac}}{2} \\ \frac{W_{1ab}^*}{2} & \Omega & 0 \\ \frac{W_{2ac}^*}{2} & 0 & \Omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

Superposition of quantum states

- General solution

$$|\psi_I(t)\rangle = \frac{\alpha}{2\hbar\Omega_0} \begin{pmatrix} 0 \\ W_{2ac} \\ -W_{1ab} \end{pmatrix} + \frac{\beta}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{W_{1ab}^*}{2\hbar\Omega_0} \\ -\frac{W_{2ac}^*}{2\hbar\Omega_0} \end{pmatrix} e^{i\Omega_0 t} + \frac{\gamma}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{W_{1ab}^*}{2\hbar\Omega_0} \\ \frac{W_{2ac}^*}{2\hbar\Omega_0} \end{pmatrix} e^{-i\Omega_0 t}$$

- Initial state

$$|\psi_I(0)\rangle = \begin{pmatrix} \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \\ \frac{W_{2ac}}{2\hbar\Omega_0} \alpha + \frac{W_{1ab}^*}{2\hbar\Omega_0} \left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \right) \\ \frac{-W_{1ab}}{2\hbar\Omega_0} \alpha + \frac{W_{2ac}^*}{2\hbar\Omega_0} \left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \right) \end{pmatrix}$$

Dark state

- When the initial state is

$$|\psi_I(0)\rangle = \begin{pmatrix} 0 \\ \sin \theta \\ \cos \theta \end{pmatrix} \quad \tan \theta = \frac{-W_{2ac}}{W_{1ab}}$$

- The state is fully at the zero frequency mode, and the state is constant of time

$$|\psi_I(t)\rangle = \begin{pmatrix} 0 \\ \sin \theta \\ \cos \theta \end{pmatrix}$$

- The state is not excited by the EM waves, and is called dark state

Electromagnetically induced transparency

- If $W_{2ac} \ll W_{1ab}$

- The initial state at $|\psi_I(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\beta + \gamma = 0$$

$$W_{2ac} \alpha + W_{1ab}^* \left(-\frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \right) = W_{2ac} \alpha - \sqrt{2} W_{1ab}^* \beta = 0$$

$$\frac{-W_{1ab} \alpha - \sqrt{2} W_{2ac}^* \beta}{2\hbar\Omega_0} = 1 \quad \alpha = \frac{W_{1ab}^*}{2\hbar\Omega_0} \quad \beta = \frac{W_{2ac}}{\sqrt{2} W_{1ab}^*} \alpha$$

- Probability at $|\alpha\rangle$

$$\langle \alpha | \psi_I(t) \rangle = \frac{\beta}{\sqrt{2}} (e^{i\Omega_0 t} - e^{-i\Omega_0 t}) = \frac{1}{\sqrt{2}} \frac{W_{2ac}}{\sqrt{2} W_{1ab}^*} \frac{W_{1ab}^*}{2\hbar\Omega_0} (2i \sin \Omega_0 t) = \frac{i W_{2ac}}{2\hbar\Omega_0} \sin \Omega_0 t$$

$$P_a = \frac{|W_{2ac}|^2}{|W_{2ac}|^2 + |W_{1ab}|^2} \sin^2 \Omega_0 t \ll 1$$