Angular momentum

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central force problem

- in 3D system

$$H = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r})$$

- μ reduced mass
- central force, potential only depends on distance

$$V(\mathbf{r}) = V(r)$$

conservation of L

- Angular momentum is conserved in central force problem

$$\frac{d\mathbf{L}}{dt} = 0$$

- In quantum mechanics, it can be written as

$$[H, \mathbf{L}] = 0$$

Introduction of L

- in classical mechanics, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- all the 3 components are

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

- because the operators are commute, the order can be changed

Introduction of L

- using p-operators

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

- In spherical coordinate

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\phi} \right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\phi} \right)$$

coordinate transformation

- the operators in spherical coordinate
- We express the components of angular momentum to be

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = x^2 + y^2 + z^2$$

$$\cos\theta = \frac{z}{r}$$

$$\tan\phi = \frac{y}{x}$$

calculation

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \cos\theta}{\partial x}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial x}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial x} = \frac{xz}{r^3\sin\theta} = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \cos\theta}{\partial y}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial y}\left(\frac{z}{r}\right) = \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial y} = \frac{yz}{r^3\sin\theta} = \frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial \theta}{\partial z} = \frac{\partial \cos\theta}{\partial z}\frac{d\theta}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = -\frac{1}{r\sin\theta} + \frac{z}{r^2\sin\theta}\frac{\partial r}{\partial z} = -\frac{x^2+y^2}{r^3\sin\theta} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \tan\phi}{\partial x}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\cos^2\phi\frac{y}{x^2} = -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \tan\phi}{\partial y}\frac{d\phi}{d\tan\phi} = \cos^2\phi\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \cos^2\phi\frac{1}{x} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial \phi}{\partial z} = 0$$

L operators in θ, φ

$$\frac{i}{\hbar}L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$= y\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) - z\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)$$

$$= y\left(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - z\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \cos^2\phi\frac{1}{x}\frac{\partial}{\partial\phi}\right)$$

$$= -\left(y\frac{\sin\theta}{r} + z\frac{\cos\theta\sin\phi}{r}\right)\frac{\partial}{\partial\theta} - \cos^2\phi\frac{z}{x}\frac{\partial}{\partial\phi}$$

$$= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$L_z = xp_y - yp_x$$

$$= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\frac{i}{\hbar}L_y = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

all these calculations can be done by sympy

$$
\begin{aligned}
\frac{i}{\hbar}L_y &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\
&= z\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) - x\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) \\
&= z\left(\frac{x}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \cos^2\phi\frac{y}{x^2}\frac{\partial}{\partial\phi}\right) - x\left(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \left(x\frac{\sin\theta}{r} + z\frac{\cos\theta\cos\phi}{r}\right)\frac{\partial}{\partial\theta} - \cos^2\phi\frac{yz}{x^2}\frac{\partial}{\partial\phi} \\
&= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}
\end{aligned}
$$

L^2

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 = \hbar^2 \left[-\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{\hbar^2 r^2} L^2$$

the angular momentum of wavefunctions

- The eigenfunctions $\psi_{nlm} = R(r)\Theta(\theta)\Phi(\varphi)$

- evaluate $L_z\psi_{nlm} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi_{nlm}$

$$L_z\psi_{nlm} = \frac{\hbar}{i}R\Theta\frac{\partial\Phi}{\partial\phi} \qquad \Phi = e^{im\varphi}$$

$$= m\hbar R\Theta\Phi$$

$$= m\hbar\psi_{nlm}$$

the angular momentum of wavefunctions

- evaluate $$L^2\psi_{nlm} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi_{nlm}$$

- recall that $$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta Y$$

$$L^2\psi_{nlm} = l(l+1)\hbar^2\psi_{nlm}$$

$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\phi)$ are also eigenfunctions of L_z and L^2

restriction on m

- We may calculate L_z^2

$$L_z^2 \psi_{nlm} = \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial \phi^2} \psi_{nlm} = m^2 \hbar^2 \psi_{nlm}$$

- For the angular momentum, it should obey

$$L^2 \geq L_z^2$$

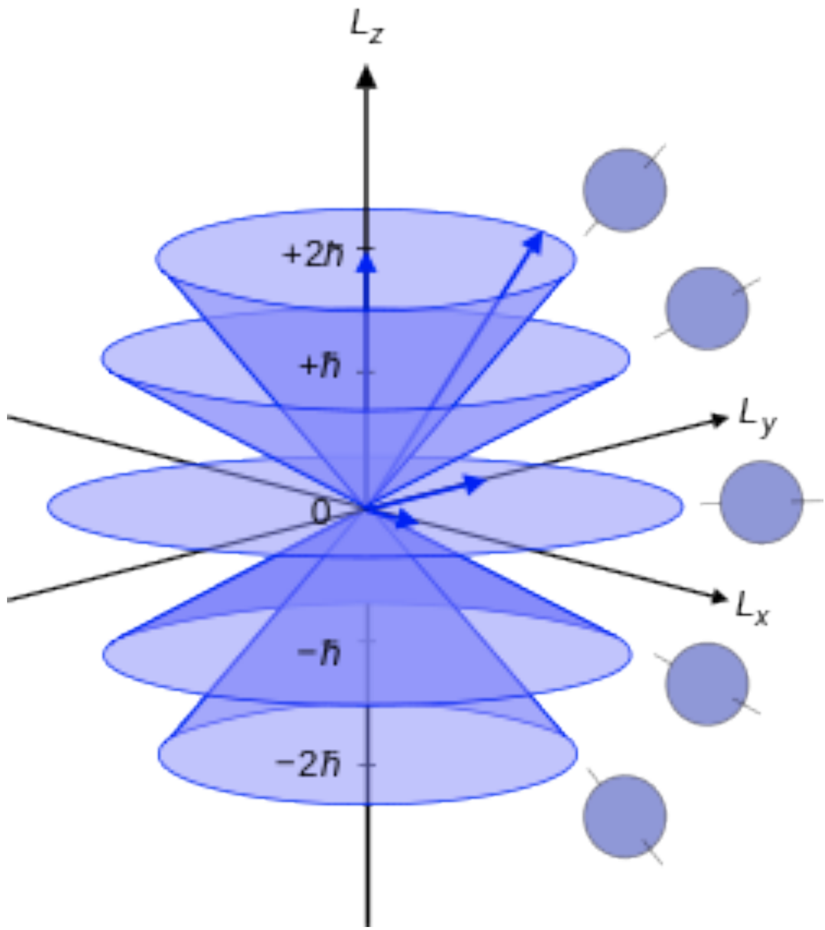
- Therefore the quantum numbers should obey

$$l(l+1) \geq m^2$$

m are integers $-l \leq m \leq l$

L vector

- Given the quantum numbers l and m, we can draw the vector in 3D (l=2 case)
- The vector cannot aligns along z-direction, namely Lx and Ly cannot be zero simultaneously. Why?



L_x and L_y

- we may check the results for other components of L

$$\frac{i}{\hbar}L_x = -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$\frac{i}{\hbar}L_y = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

- Y_{lm} are not eigenstates of these two operators

$$L_x\psi_{nlm} \neq (\text{cosntant})\psi_{nlm}$$

$$L_y\psi_{nlm} \neq (\text{cosntant})\psi_{nlm}$$