Quantum Mechanics I

Ver. Sep 19

Fundamental concepts

- Complementarity
- Superposition

let’s consider these properties of the light

Wave-particle duality

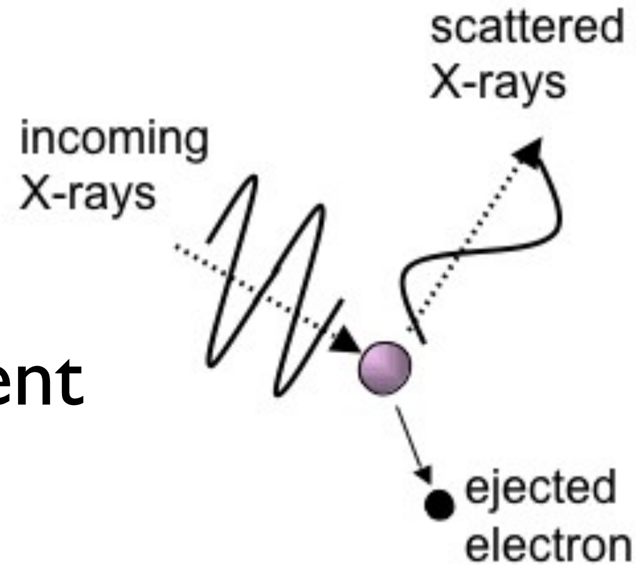
- Einstein's photon concept

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}$$

$$\hbar = 1.054 \times 10^{-34}\,\mathrm{J\ s}$$

Planck's constant

Confirmed by Compton's experiment



Complementarity concept

Compton's
radioactive source

states of the electromagnetic field with a definite number of photons, but the field strengths do not have definite values.

PARTICLE

ideal laser

states have well-defined field strengths, but not a definite number of photons.

WAVE

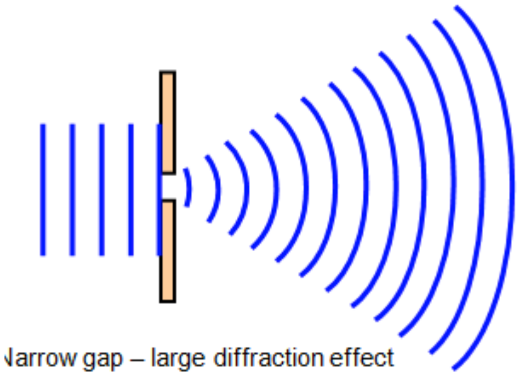
interference vs. path

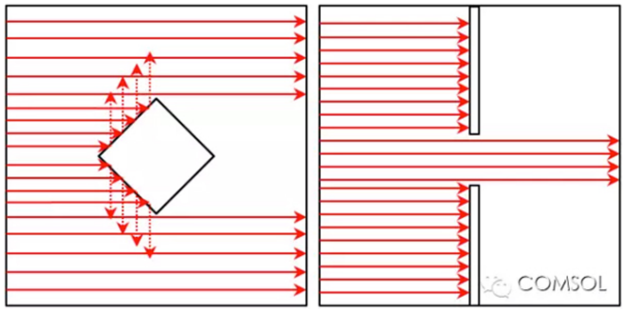
INTERFERENCE

DETERMINISTIC PATH

- In any setup that allows light to traverse different paths, these paths can either be combined coherently to form an interference pattern

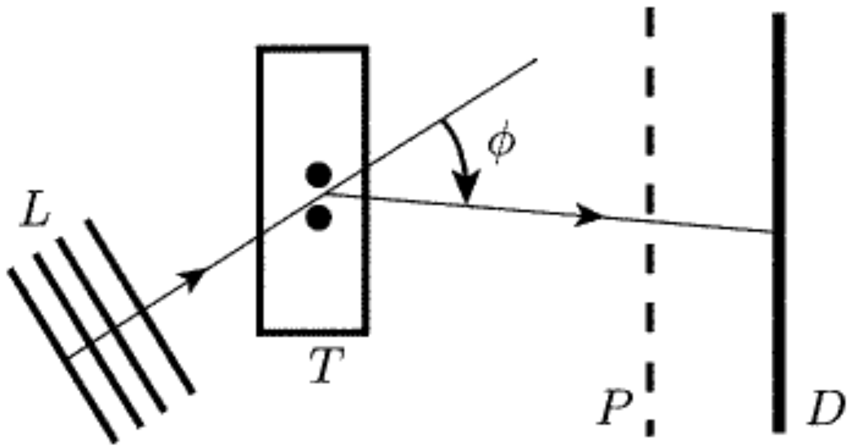
- the apparatus can be modified to determine which path is followed but this destroys the interference pattern.

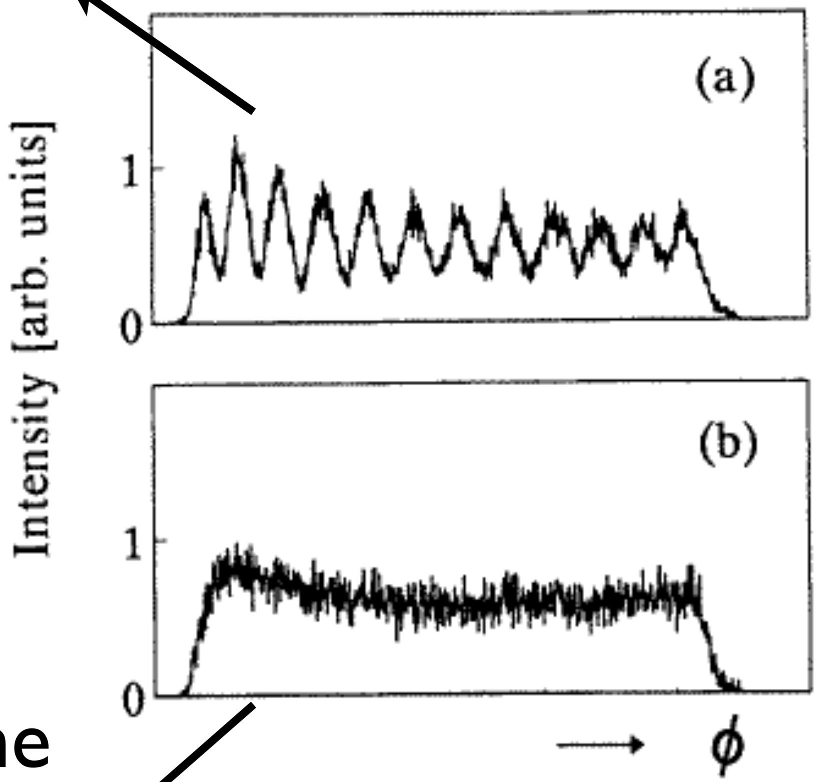




Now we look on one example

Events in which spin flip is excluded and the two possible photon paths are indistinguishable.





Events in which one of the ions must have had a spin flip which determines the photon path.

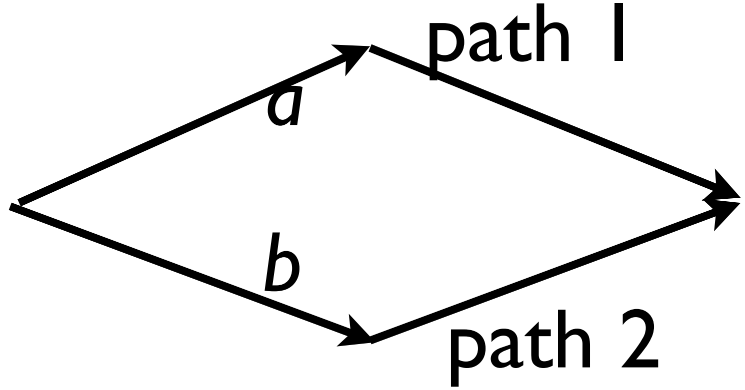
U. Eichmann, et al.
Phys. Rev. Lett. 70, 2359 (1993).

- The complementarity is resulted from simple mathematics

visibility of the interference

$$\psi = ae^{ikL_1} + be^{ikL_2}$$

$$\left|\psi\right|^2 = a^2 + b^2 + 2ab\cos\left(k\Delta L\right)$$



The visibility of the interference

$$V = \frac{\left|\psi_{\max}\right|^2 - \left|\psi_{\min}\right|^2}{\left|\psi_{\max}\right|^2 - \left|\psi_{\min}\right|^2} = \frac{2ab}{a^2 + b^2}$$

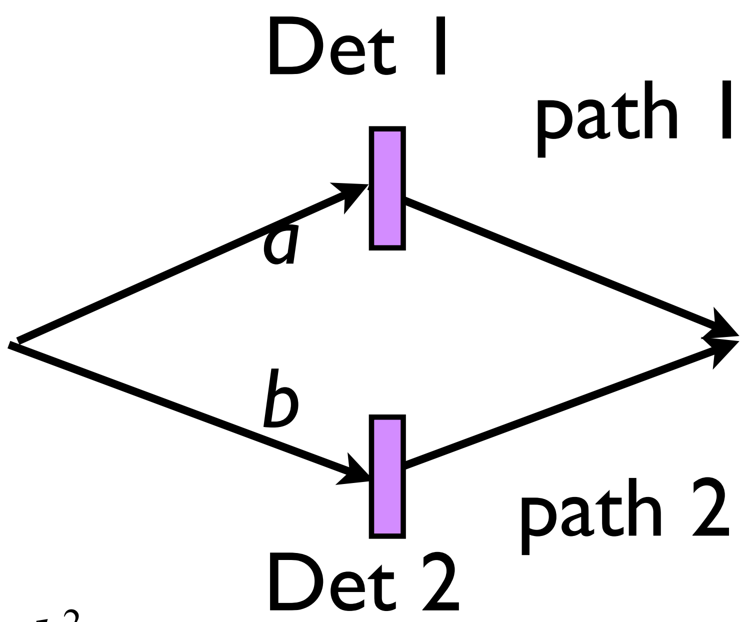
determination of path

- two detectors are placed just behind the holes
- they will register with rates proportional to

$$a^2, \quad b^2$$

- the difference in probability

$$\Delta = \frac{a^2 - b^2}{a^2 + b^2}$$



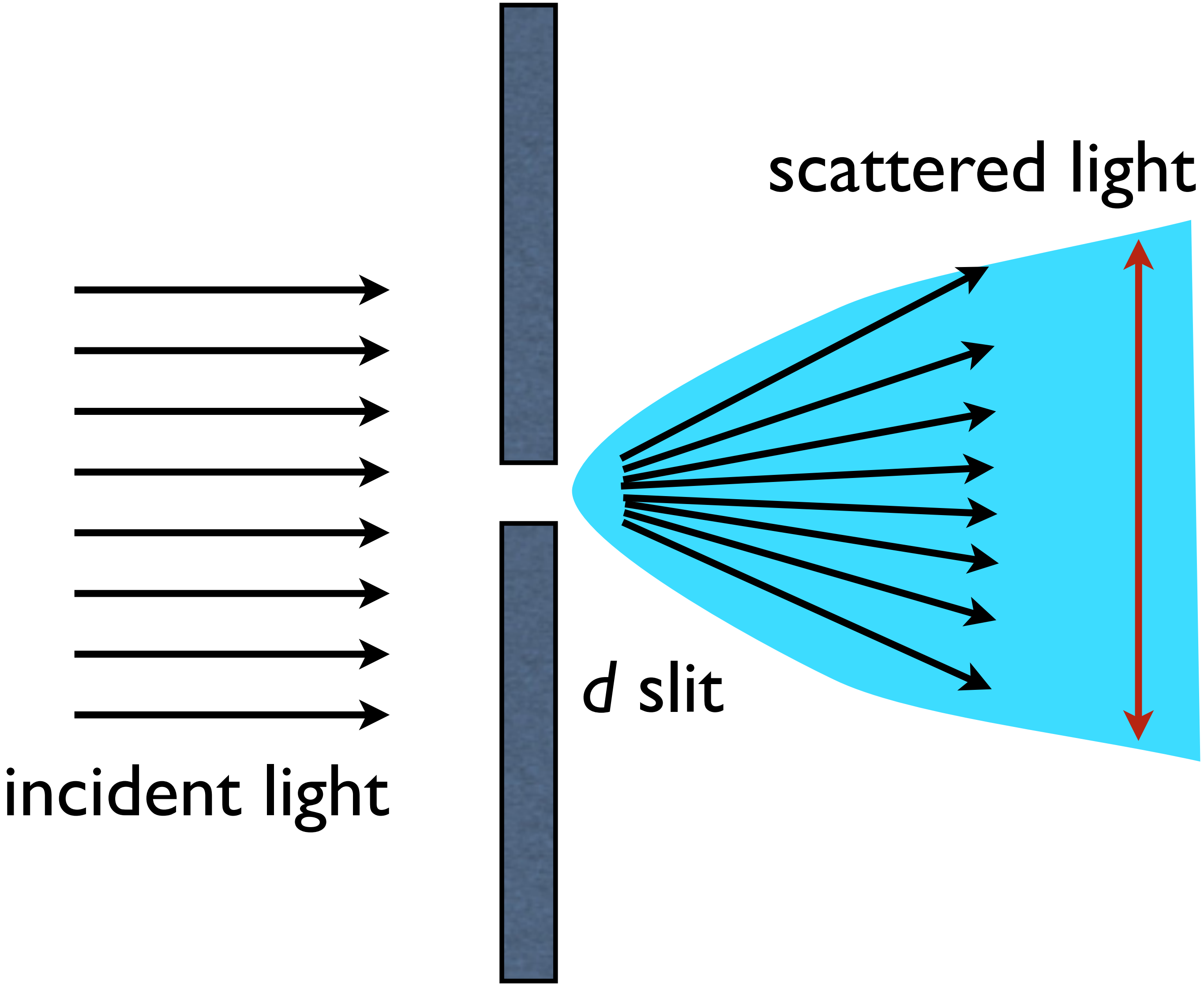
visibility of interference vs. determination of path

$$V^2 + \Delta^2 = 1$$

Higher Δ, lower V

- More math: Fourier transform

diffraction of a plane wave



diffraction angle

$$= \frac{\Delta k}{k} \sim \frac{\lambda}{d}$$

$$\Delta x \sim d$$

$$\Delta x \Delta k \sim k\lambda \sim 1$$

Fourier transform: plane waves

- wave function in x-space

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk \psi(k,t) e^{ikx} = e^{i(k_0 x - \omega t)} \qquad \Delta x \to \infty$$

- wave function in k-space

$$\psi(k,t) = \frac{1}{\sqrt{2\pi}} \int dx \psi(x,t) e^{-ikx} = \sqrt{2\pi}\,\delta(k - k_0) \qquad \Delta k \to 0$$

Gaussian wavepacket

- wave function in x-space

$$\psi(x, t=0) = A e^{ik_0 x} e^{-(x-x_0)^2/4\Delta x^2}$$

$$\psi^*(x, t=0)\psi(x, t=0) = |\psi(x, t=0)|^2 = A^2 e^{-(x-x_0)^2/2\Delta x^2}$$

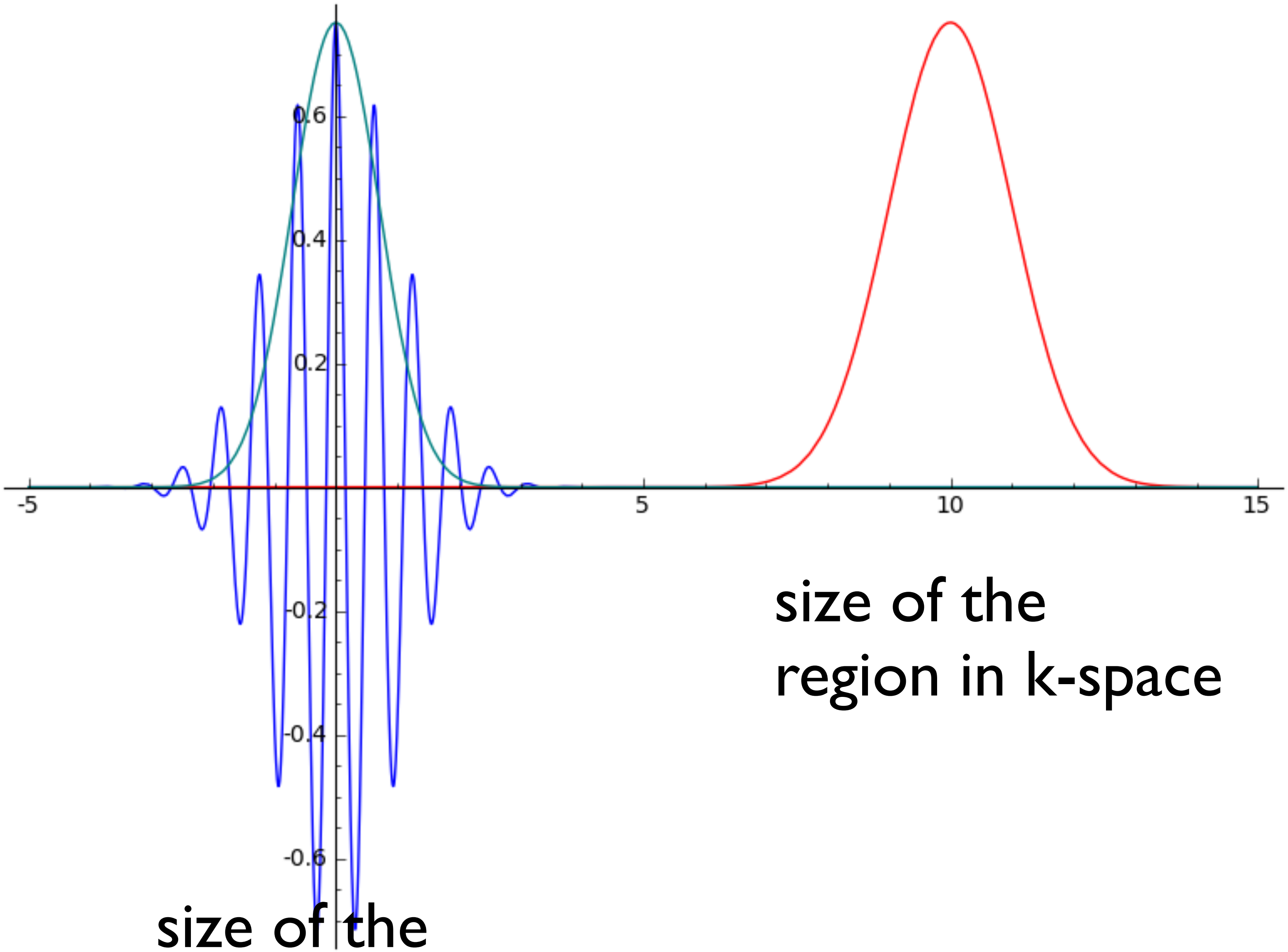
- wave function in k-space

$$\psi(k, t=0) = \frac{1}{A\sqrt{\pi}} e^{-i(k-k_0)x} e^{-(k-k_0)^2\Delta x^2}$$

- intensity(probability density)

$$|\psi(k, t=0)|^2 = \frac{1}{A^2\pi} e^{-(k-k_0)^2 2\Delta x^2} = \frac{1}{A^2\pi} e^{-(k-k_0)^2/2\Delta k^2}$$

$$\Delta k \Delta x = \frac{1}{2}$$



size of the
region in k-space

size of the
spatial region

Uncertainty principle

- The classical electric field

$$\mathbf{E}(\mathbf{r},t) = \int d\mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{a}(\mathbf{k}) \qquad \omega = ck, \quad \mathbf{k}\cdot\mathbf{a} = 0$$

- The theory of Fourier integrals tells us that the size of the region in k-space in which the Fourier amplitude a(k) is substantial is related to the size of the spatial region

$$\Delta x_i \Delta k_j \geq \delta_{ij}$$

- the time that the packet takes to pass any point is related to the dispersion in frequency

$$\Delta t \Delta \omega \geq 1$$

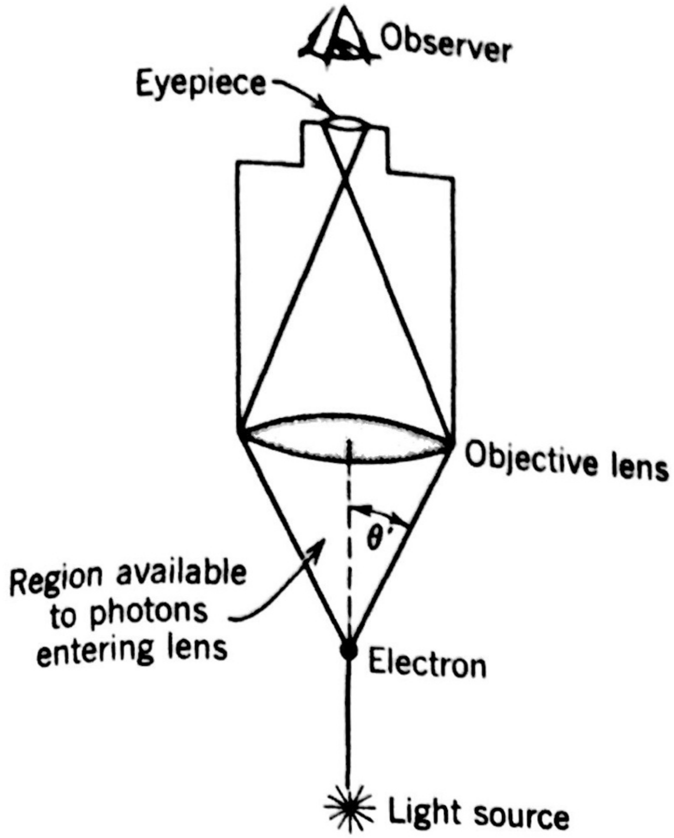
- In physics, k is not merely a math object, but has physical meanings. It is the momentum of the photon.

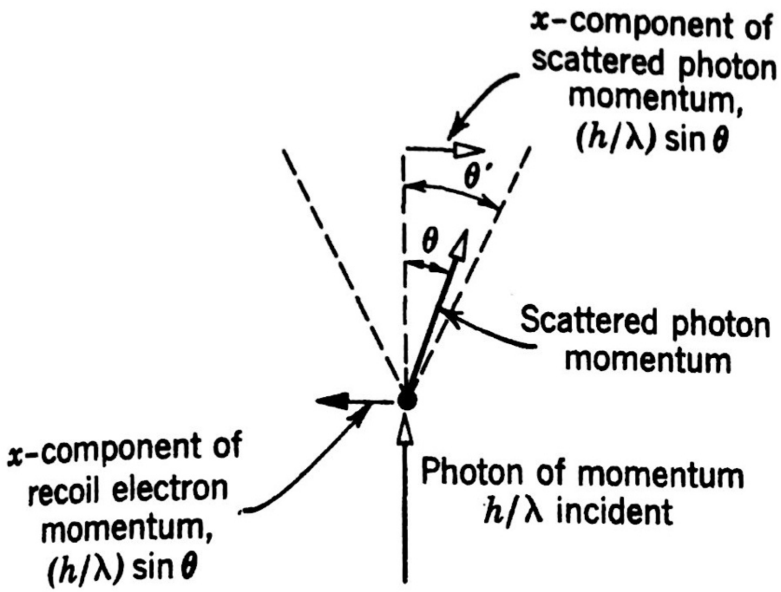
Heisenberg uncertainty relations

For photons $\Delta x_i \Delta p_j \geq \hbar \delta_{ij}$ $\Delta E \Delta t \geq \hbar$

- Once the ability to determine any object's momentum and energy is restricted by the uncertainty relations, those of all other objects with which it can, in principle, interact must also satisfy such restriction.(For example, to measure electron using light waves)
- Energy and momentum are conserved by any isolated systems on an event-by-event basis.

Bohr’s model of microscope

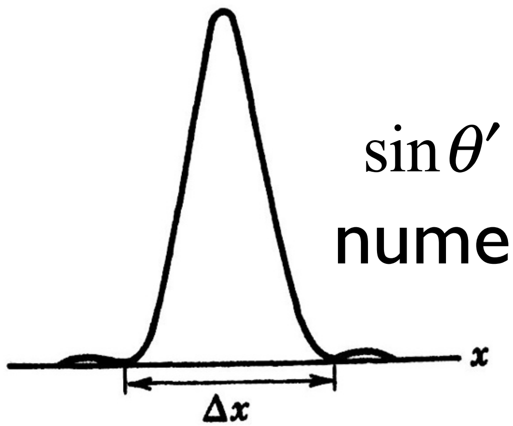




the electron recoiled by the photon

$$\Delta p_x = 2p\sin\theta'$$

scattered photon has the diffraction limit $\Delta x = \lambda/\sin\theta'$



$\sin\theta'$ is called numerical aperture

massive particles

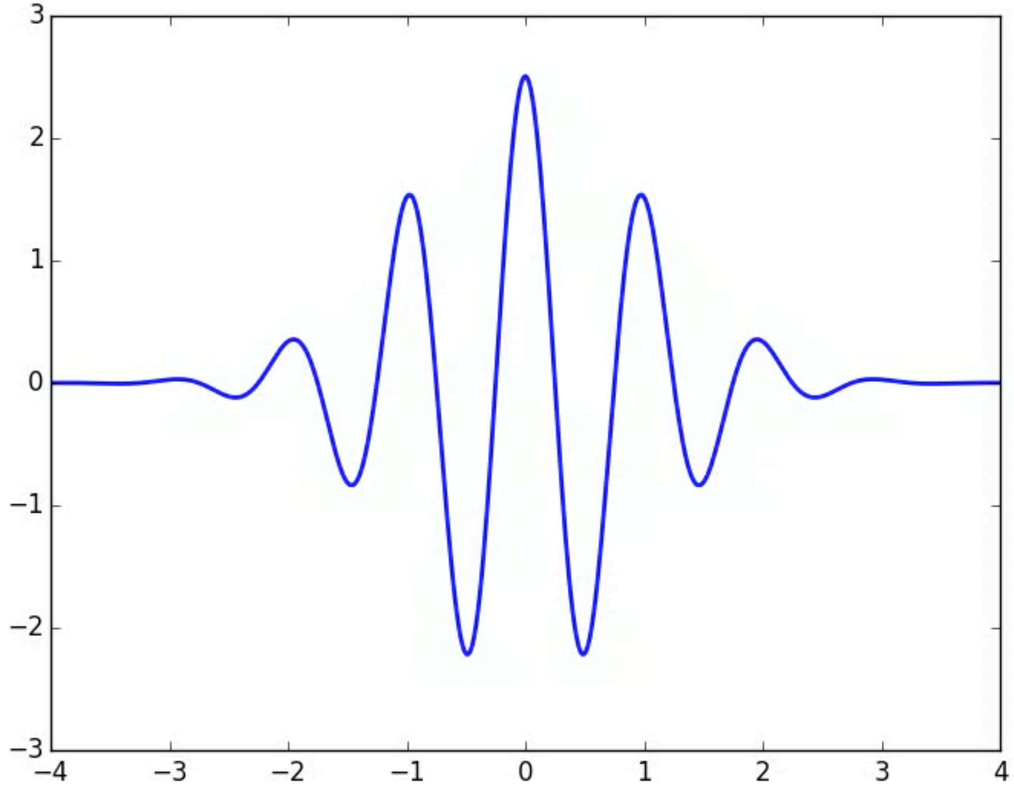
- de Broglie wavelength $\lambda = \dfrac{h}{p}$

- dispersion relation

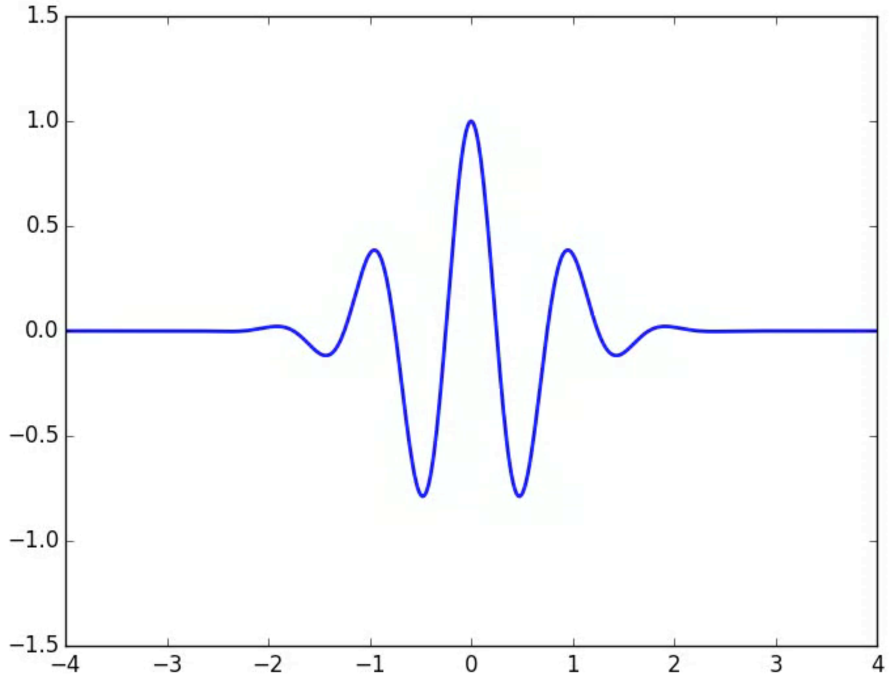
$$E = \frac{\mathbf{p}^2}{2m}$$

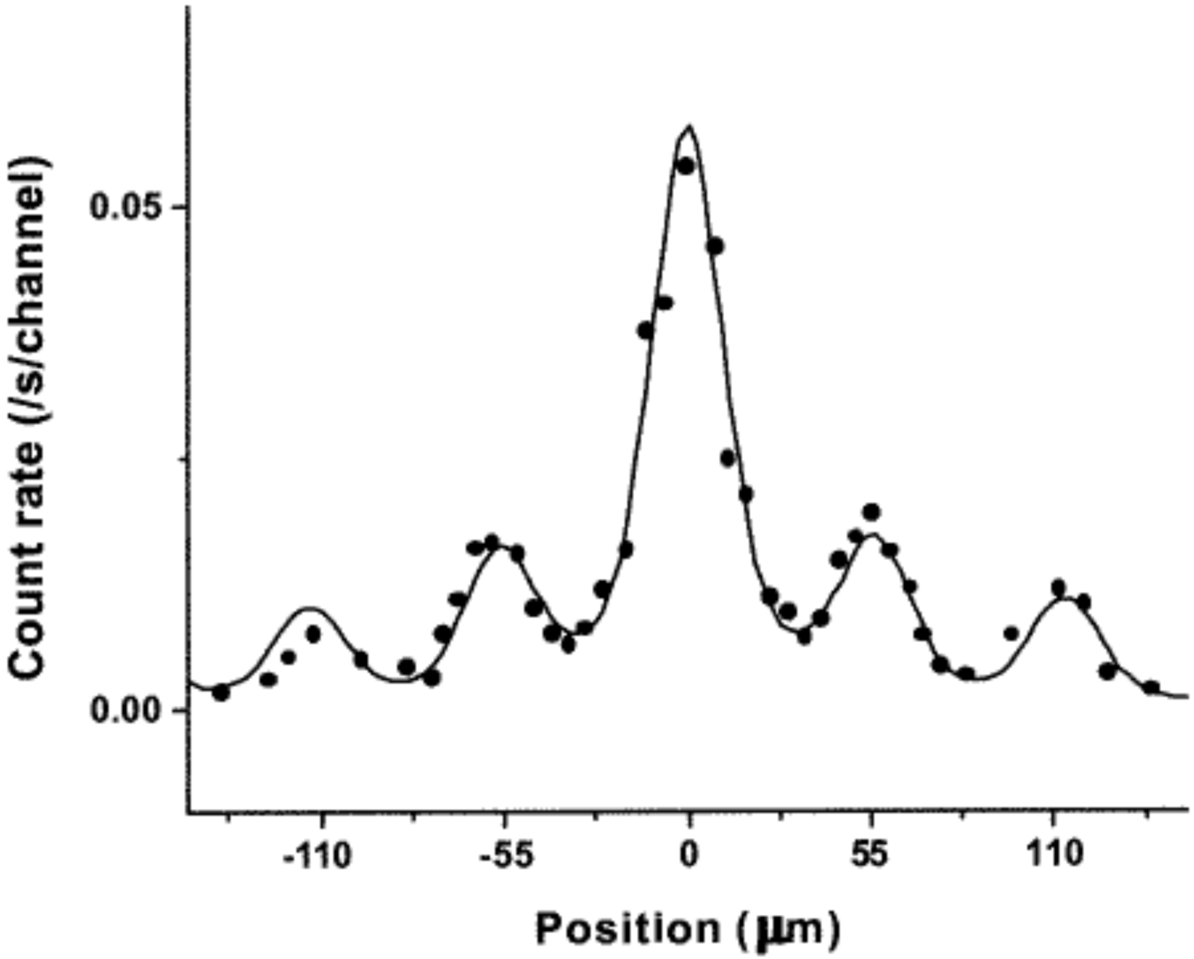
$$\Longrightarrow \quad \hbar\omega(\mathbf{k}) = \frac{\hbar^2 k^2}{2m}$$

non-linear dispersion



linear dispersion



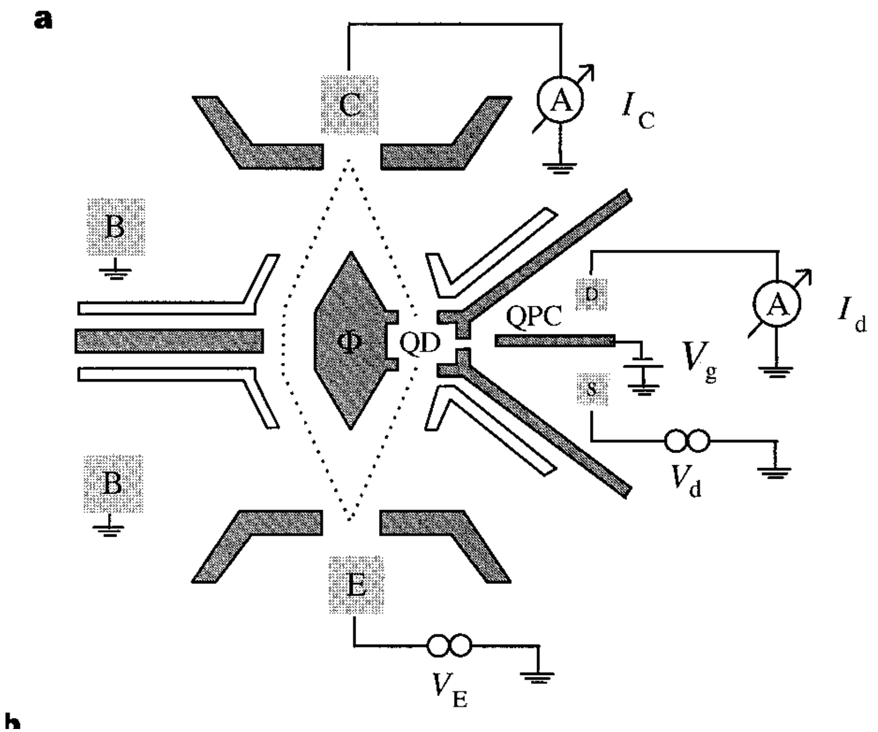


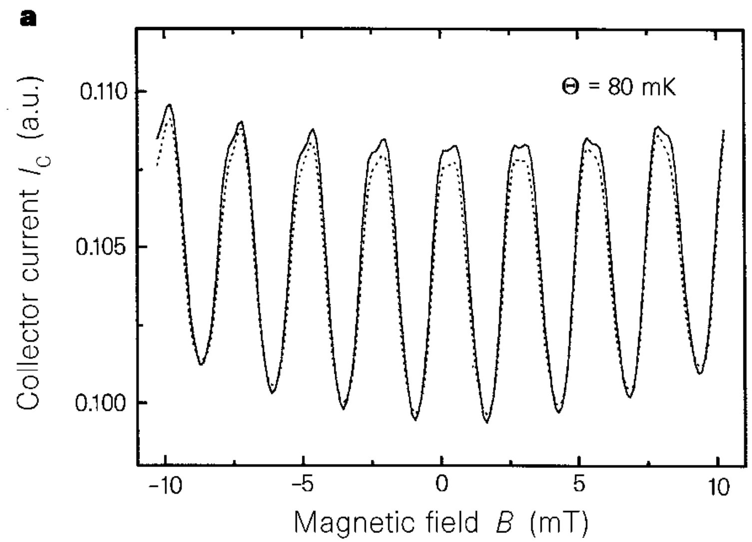
Diffraction pattern produced by scattering electrons from the standing light wave created by two opposed lasers.

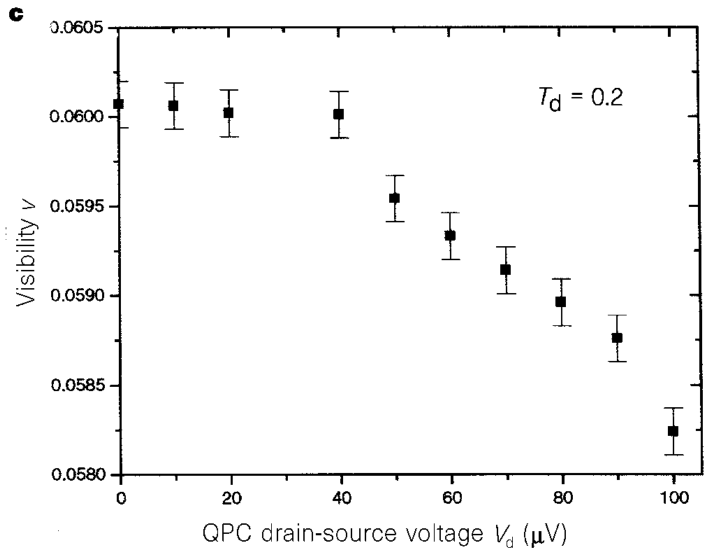
D.L. Freimund, K. Afiatooni and H. Batelaan,
Nature 413, 142 (2001)

"Which path" experiment

- by varying the sensitivity of the detector the visibility of the oscillatory interference signal is affected







E. Buks, et al , Nature **391**, 871 (1998)

Superposition principle

- Quantum mechanics is a strictly linear theory
- Schrodinger's equation

$$\left(H - i\hbar \frac{\partial}{\partial t} \right) \psi\left(t\right) = 0$$

- the linear superposition of any two solutions is a solution

$$\psi = c_1 \psi_1 + c_2 \psi_2$$

continuous linear superposition

In general, any wave can be expressed as a linear combination of plane waves using amplitude function $A(\mathbf{k})$

$$\Psi\left(\mathbf{r},t\right)=\sum_{\mathbf{k}}A\left(\mathbf{k}\right)e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)}$$

The integral form:

$$\Rightarrow \quad \Psi\left(\mathbf{r},t\right)=\iiint A\left(\mathbf{k}\right)e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)}d^3k$$

The completeness and uniqueness of the above expression need further proof. One can refer to Fourier's theorem

Quiz 1-1

- The mass of electron is 0.51MeV (or 9.1×10^{-31} kg). Find out the de Broglie wave length for a non-relativitic electron (namely E<<0.15MeV) with energy of E in unit of eV.

- Compare the result to the photons.

Quiz 1-2

- Estimate the photon number in the volume of cubic wavelength in the sunlight beam. The energy flux $S=10^3 W/m^2$

- Do the same estimation on electron beams. Acceleration voltage =30keV, beam current =10pA, beam spot diameter=10nm.