Quantum mechanics problem set 4

December 28, 2020

1. Consider the spin state
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
in which the eigenstates can be written as
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
(a) Write the corresponding density matrix for a pure state $\rho_\varphi = |\psi\rangle\langle\psi|$
(b) Calculate $tr(\rho_\varphi)$ and $tr\big(\rho_\varphi^2\big)$.

2. Another density matrix describes a mixture
$$\rho_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$
(a) Calculate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ for ρ_ψ and ρ_2. Here $\sigma_{x,y,z}$ are Pauli matrices.
(b) Could we distinguish the difference of a pure state and a mixture by measurement?

3. Proof the identities:
(a) $[p^n, q] = -in\hbar p^{n-1}$
(b) $\left[e^{\frac{iap}{\hbar}}, q\right] = ae^{\frac{iap}{\hbar}}$

4. Proof the identities:
(a) $\langle q'|p^n|\psi\rangle = (\frac{\hbar}{i})^n \frac{d^n}{dq'^n}\langle q'|\psi\rangle = (\frac{\hbar}{i})^n \frac{d^n}{dq'^n}\psi(q')$
(b) $\langle q'|e^{\frac{iap}{\hbar}}|\psi\rangle = \psi(q'+a)$

5. Show that for any hermitian operator A, the eigenvalues are not changed under the unitary transform $U^\dagger AU$.

6. Consider the spin hamiltonian $H = -\hbar\omega_0\sigma_z$, and try to write down the analytical form of time evolution operator $U(t)$.
(a) If the spin is initially kept at state $|\psi, t=0\rangle = |x+\rangle$, calculate the state $|\psi, t\rangle$ at any time t. Describe how the spin changes in time.
(b) Calculate x measurement result $\langle\sigma_x\rangle$ as a function of time.

7. Use a program (e.g. python) to calculate the spin evolution for any hamiltonian and any initial state. Verify your answer with the previous problem.

8. Following the above question, calculate how the spin operators σ_x and σ_z change in time in the Heisenberg picture.