Quantum mechanics problem set 2

October 26, 2020

1. Suppose the dimension of a Hilbert space is d, and also U is a $d \times d$ matrix obeying $U^\dagger U = 1$. With an orthonormal basis $|k\rangle$ in the Hilbert space, show that the set $U|k\rangle$ is also an orthonormal basis. Use this fact to prove that $UU^\dagger = 1$, so that U must be unitary.

2. There is an orthonormal basis set $\{|n\rangle\} = |1\rangle, |2\rangle, |3\rangle, \cdots$ for an infinite dimensional Hilbert space. (a) An operator U is defined by

$$U = \sum_{n=1}^{\infty} |n+1\rangle \langle n| \,.$$

   Show that $U^\dagger U = 1$, but $UU^\dagger \neq 1$. Thus U is not unitary. (b) Could you fix this problem to suggest a unitary operator in a finite dimensional Hilbert space?

3. Consider a 3-dimensional ket space. By using an orthonormal set $|1\rangle\,, |2\rangle$ and $|3\rangle$, the operators A and B are represented as

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}.$$

   Here a and b are real numbers.
   (a) What are the eigenvalues of A and B?
   (b) Show that A and B are commute.
   (c) Find the new orthonormal kets that are simultaneous eigenkets of A and B. Specify the eigenvalues of A and B for each of the 3 eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

4. For an input state

$$|\phi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

   What are the joint probabilities for the all possible outcome of simultaneous measurement A and B, which are defined in previous problem?

5. If H is an hermitian operator, show that (a)$U_1 = e^{iH}$ (b)

$$U_2 = \frac{1 + iH}{1 - iH}$$

are unitary operators.
(c) Let

$$H = \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix},$$

calculate U_1 and U_2 explicitly.

6. Prove that the eigenvectors associated to different eigenvalues of an operator are orthogonal.