Quantum mechanics problem set 1

September 24, 2020

1. What are the wavelengths of a photon and an electron with a given energy? Use both non-relativistic and relativistic energy-momentum relations in the electron case. Next, do numerical calculations and illustrate these results in a log-log plot by using the energy ranging from 1eV to 10 GeV. Could you comment the results?

2. A single photon source generates the photon density at most one photon per cubic wavelength. For a green light of 500 nm wavelength, how large is the energy flux (power per unit area) such a photon source emit?

3. The kinetic energy K of a particle is related to its momentum p by $K = p^2/2\mu$, where μ is the particle's mass. In a gas at absolute temperature T, the molecules have a typical kinetic energy of $3k_BT/2$, in which k_B is Boltzmann constant. Derive an expression for the thermal de Broglie wavelength, a typical value for the de Broglie wavelength λ of a molecule in a gas. For helium atoms ($\mu = 6.7 \times 10^{-27}$ kg), calculate the thermal de Broglie wavelength at room temperature (T= 300 K) and at the boiling point of helium (T= 4 K). Quantum effects become most significant in matter when the thermal de Broglie wavelength of the particles is greater than their separation. At atmospheric pressure, gas molecules are about 1-2 nm apart; in a condensed phase (liquid, solid) they are about ten times closer. How do these compare with the thermal de Broglie wavelengths you calculated for helium?

4. Calculate and plot the wavefunction in k-space of a Gaussian pluse function
$$\phi(x, t = 0) = Ae^{ik_0x} \exp\left(-\frac{x^2}{4\Delta x^2}\right).$$
in which Δx and k_0 are non-zero constants. You may assume the values on your own or set them as variables in your calculations. Plot the time-evolved wavefunction $\phi(x, t)$ with a non-linear dispersion relation, $\omega = \hbar k^2/2m$. You may apply one of the methods: (1) Do the analytical calculation and plot the results using any software. (2) Numerically calculate and plot the results using any software.

5. The orthnormal basis defined in $x = [0, L]$ can be sinusoidal functions

$$u_a(x) = \frac{1}{\sqrt{L}} \exp\left(\frac{i2a\pi x}{L}\right) \quad \text{for} \quad a = 0, \pm 1, \pm 2 \cdots .$$

Demonstrate that they are orthnormal in $x = [0, L]$, namely

$$\int_0^L u_a^*(x)\, u_{a'}(x)\, dx = \delta_{aa'}.$$

6. Any real function (x) defined in $x = [0, L]$ can be superposed by the basis we introduced in the last problem, namely

$$f(x) = \sum_a c_a u_a(x).$$

Assume a polynomial function on your own. Use any software to calculate the first 11 coefficients and plot the original function and the superposed function using the first 10 series.

7. The completeness of this basis requires that

$$\sum_a u_a(x)\, u_a(y) = \delta(x - y).$$

Do one of the following tasks to demonstrate the result: (1) Prove the expression analytically, or (2) numerically calculate the expression by using any software to sum finite terms.