# Quantum mechanics problem set 3

December 24, 2019

1. The hamiltonian of a spin in magnetic field $\vec{B}$ is $H = -\gamma \vec{S} \cdot \vec{B}$. $\gamma$ is gyromagntic ratio, and vector $\vec{S}$ is the angular momentum operator, $\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$, expressed by Pauli matrices, $\sigma_i$ . For example, if a magnetic field is pointing to $x$-direction, the hamiltonian reads $H = -\frac{\gamma \hbar B}{2}\sigma_x$.

   Now a magnetic field points to a direction which can be described by the polar angle $\theta$ and azimuthal angle $\phi$. Namely,

   $$H = -\frac{\gamma \hbar B}{2}\left(\cos\theta\sigma_z + \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y\right)$$

   Find the energy eigenvalues and corresponding eigenkets using $|+z\rangle$ and $|-z\rangle$

2. (a) Calculate the commutation relations of the Pauli matrices: $[\sigma_x, \sigma_y] =?$, $[\sigma_y, \sigma_z] =?$, and $[\sigma_z, \sigma_x] =?$
   (b) Calculate the anti commutation relations: $\{\sigma_x, \sigma_x\} =?$, $\{\sigma_x, \sigma_y\} =?$, and $\{\sigma_x, \sigma_z\} =?$

3. (a) Find out the $2 \times 2$ matrices of $\sqrt{\sigma_x}$, $\sqrt{\sigma_y}$, and $\sqrt{\sigma_z}$.
   (b) Show that the three matrices respectively represent the $\pi/2$ rotation about $x$, $y$ and $z$ axis for a quantum state in the Bloch sphere.

4. The spin-up and spin-down states are expressed by $|0\rangle$ and $|1\rangle$. One of the Bell states for two spins is written as

   $$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\right).$$

   Starting with the state $|0\rangle_1|0\rangle_2$, how can one get this Bell state? (Hint: you may use single qubit gates and controlled-NOT gate)

5. Following the previous problem, if the spins are distributed to Alice and Bob. After Alice performs the measurement of $S_z$ and gets the value of $\hbar/2$,
   (a) how much probability for Bob observing $\hbar/2$ if he does the same measurement $S_z$?

(b) how much probability for Bob observing $\hbar/2$ if he does the measurement $S_x$?
(c) how much probability for Bob observing $\hbar/2$ if he does the measurement $(S_x + S_z)/\sqrt{2}$?